LECTURE 14: BERNOULLI OR BINARY SHIFT MAP

Consider the logistic map when $\mu = 4$:

$$x_{n+1} = 4x_n(1 - x_n).$$

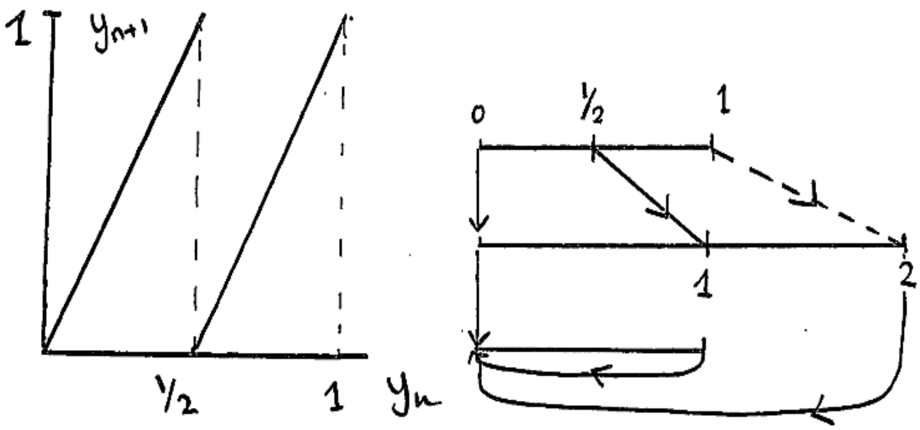
Let $x_n = \sin^2(\pi y_n)$. Then

$$\sin^2(\pi y_{n+1}) = 4\sin^2(\pi y_n)(1 - \sin^2(\pi y_n)) = \sin^2(2\pi y_n), \qquad (2.1)$$

$\forall \quad 0 \leq y_n \leq 1$. (2.1) is equivalent to

$$y_{n+1} = 2y_n \bmod 1 = \begin{cases} 2y_n, & 0 \leq y_n < 1/2, \\ 2y_n - 1, & 1/2 \leq y_n < 1. \end{cases} \qquad (2.\!:$$

This is the Bernoulli or Binary Shift Map, with a simple discontinuity at $y_n = 1/2$:



Consider successive iterates:

$$y_1 = 2y_0 \bmod 1, y_2 = 2y_1 = 2^2 y_0 \bmod 1, \ldots, y_n = 2^n y_0 \bmod 1.$$

Therefore any point in $0 \leq y < 1$ has a binary expansion:

$$y_0 = \frac{a_1}{2} + \frac{a_2}{2^2} + \ldots + \frac{a_n}{2^n} + \ldots = 0.a_1 a_2 a_3 a_4 \ldots a_n,$$

for $a_k \in \{0,1\}$. Then

$$y_1 = 2y_0 = a_1 + \frac{a_2}{2} + \ldots = a_1 + \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i} = a_1.a_2a_3a_4,$$

ie. iterations move the decimal point one place to the right and delete terms to the left.

NB. After enough iterations, we lose all information in i.c. y_0.

This is an example of the ‘sensitive dependence on i.c.s’, one of the key features of chaotic systems:

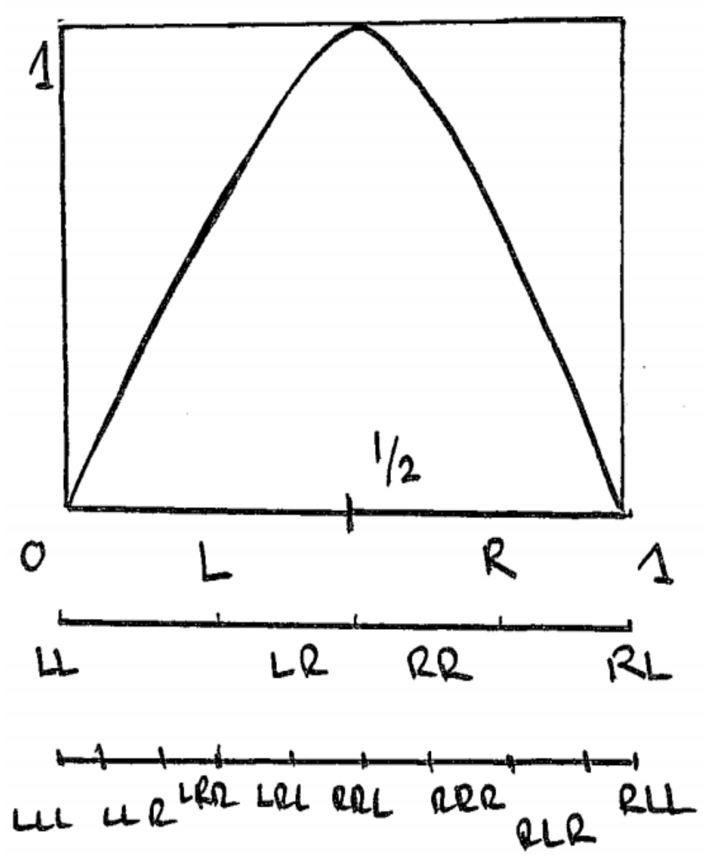
Example If $y_0 = \cdot a_1a_2 \ldots a_n b \ldots$ and $z_0 = \cdot a_1a_2 \ldots a_n c \ldots$ with $b \neq c$. then after n iterations, $y_n = \cdot b \ldots$; $z_n = \cdot c \ldots$ and $y_n \neq z_n$.

2.1 Symbolic Dynamics

Instead of using $\{0,1\}$, we can use $\{L,R\}$ to describe iterations for the logistic map

An itinerary of an orbit is a sequence of symbols: $L^iR^jL^kR^m$.

For the logistic map for $\mu = 4$, the first symbol is L if $x \in [1, 1/2]$ and R if $x \in [1/2, 1]$



Rule for iterations and sub dimensions

1. Count the number of Rs in the sequence.

2. If the number of R is even, the sequence for the left subinterval of the next iteration ends in L and that of the right subinterval ends in R.

3. If the number of R is odd, we interchange the rule in (b) above.

Special orbit: $x = 1/2$ has the orbit $\{1/2, 1, 0, 0, \ldots\}$ with the sequence R^2L^n or LRL^n.
When $\mu = 4$, the logistic map displays all the characteristics of chaos:

1. Countable infinity of periodic orbits (associated with the rational numbers).

2. Uncountable infinity of aperiodic orbits (associated with the irrationals or the reals).

3. Sensitive dependence on i.c.s.

4. A dense orbit, which comes arbitrarily close to all periodic orbits.

2.2 The Skinny Baker Map

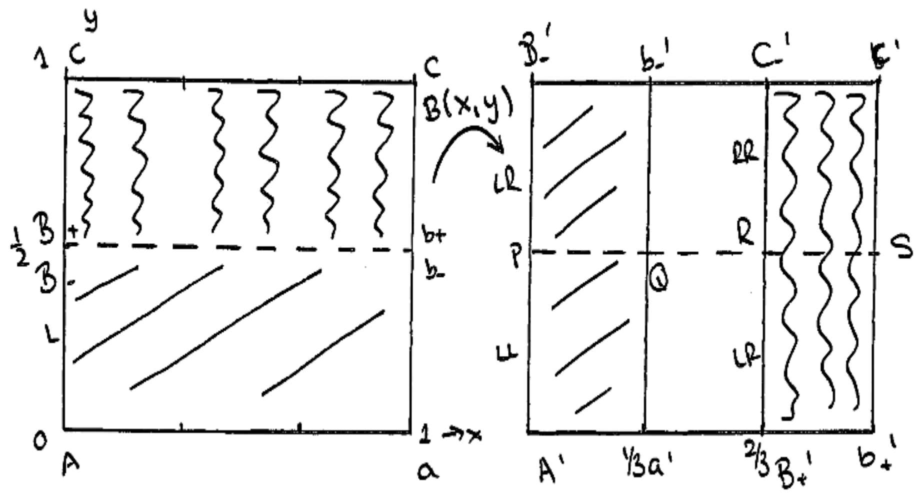
As a prelude to discussing the Smale Horseshoe map, we consider the Skinny Baker map
This exhibits the features of stretching in one direction and shrinking in the other- properties of a linear 2-D map but with re-injection as in the Smale Horseshoe map.
The map is defined on the unit square:

$$B(x,y)=\begin{cases}(1/3x,2y) & 0\le y\le 1/2,\quad 0<x\le 1,\\ \\ (1/3x+2/3,2y-1) & 1/2<y\le 1,\quad 0<x\le 1,\end{cases}$$

or

$$B(x,y)=\begin{cases}\begin{bmatrix}1/3 & 0\\ \\ 0 & 2\end{bmatrix}\begin{bmatrix}x\\ \\ y\end{bmatrix} & \text{for}\quad 0\le y\le 1/2,\\ \begin{bmatrix}1/2 & 0\\ \\ 0 & 2\end{bmatrix}\begin{bmatrix}x\\ \\ y\end{bmatrix}+\begin{bmatrix}2/3\\ \\ -1\end{bmatrix} & \text{for}\ 1/2<y\le 1.\end{cases}$$

Consider its action on the unit square, S:



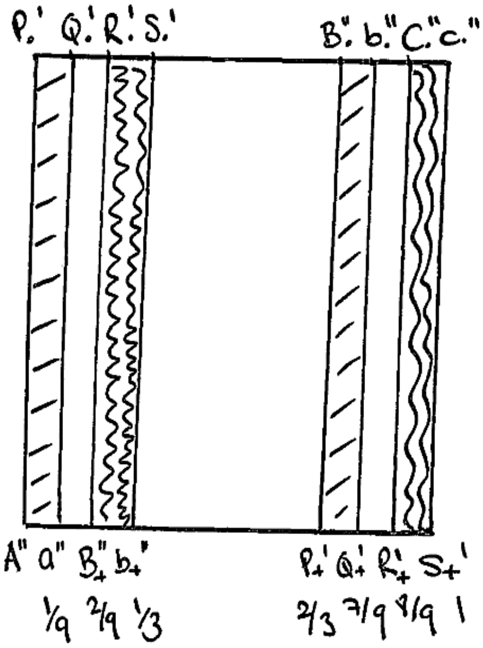
Under B the map contracts in the x direction by 1/3 and expands in the y direction by 2, discarding the middle 1/3 rectangle, while preserving orientation,

e.g.:

$$\begin{bmatrix}x_{a'}\\ \\ y_{a'}\end{bmatrix}=\begin{bmatrix}1/3 & 0\\ \\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ \\ 0\end{bmatrix}\rightarrow\begin{bmatrix}1/3\\ \\ 0\end{bmatrix};$$

$$\begin{bmatrix} x_{c'} \\ y_{c'} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

The second iterate is:



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$$\begin{bmatrix} x_{B''_+} \\ y_{B''_+} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2/9 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} x_{B''_-} \\ y_{B''_-} \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Definition 1. *The invariant set of the map B is*

$$(x, y) : (x, y) \in B^n \quad \forall n > 0 \quad \textit{and} \quad n < 0,$$

since the inverse map B^{-1} is defined as:

$$B^{-1}(X, Y) = \begin{cases} (3X, Y/2) & 0 \le X \le 1/3, \quad 0 < Y \le 1, \\ (3X - 2, Y/2 + 1/2) & 2/3 < X \le 1, \quad 0 < Y \le 1. \end{cases}$$

Definition 2. *The shift map S is defined on the space of 2-sided iterates by*

$$s(\dots S_{-3}S_{-2}S_{-1}S_0.S_1S_2S_3\dots) = \dots S_{-2}S_{-1}S_0S_1.S_2S_3\dots$$

NB: For all points lying in the invariant set of the unit square, the iterates are bi-infinite strings of forward and backward images under B. Backward images involve the application of the inverse map B^{-1} and shift the symbol sequence one place to the left, while forward images, involving application of B, shift the symbol sequence one place to the right.

Thus

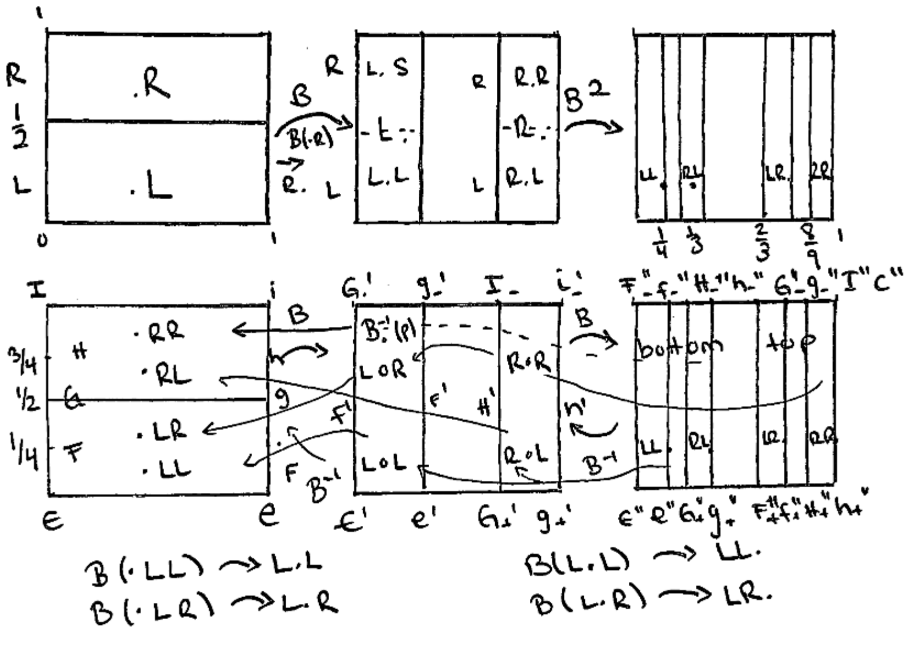
$$B(\dots S_{-2}S_{-1}S_0.S_1S_2S_3\dots) = \dots S_{-2}S_{-1}S_0S_1.S_2S_3\dots$$

and

$$B^{-1}(\dots S_{-2}S_{-1}S_0.S_1S_2S_3\dots) = \dots S_{-2}S_{-1}.S_0S_1S_2S_3\dots.$$

NB: There are no deletions of symbols, c.f. the Bernoulli shift (which is non-invertible).

We illustrate this by returning to the Baker's map: labelling the bottom half by L and the top half by R.



Thus $P \in LR\dots, \Rightarrow B^{-1}(p) \in L.R$ and $B^{-2}(p) \in .LR$ and $q \in .RL \Rightarrow B(q) \in R.L$ and $B^2(q) \in RL$.