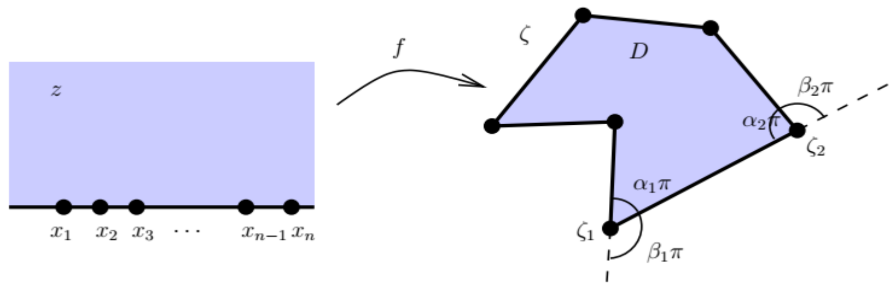
Lecture 4: Schwarz-Christoffel

A (rare) constructive method for finding conformal maps (as opposed to cataloguing them) is the Schwarz-Christoffel formula. This lets us map a half-plane to a polygon (and there is an extension to circular polygons), and hence the inverse maps a polygon to a half-plane.

Our target domain is a polygon with interior angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$, at the vertices $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_n$.



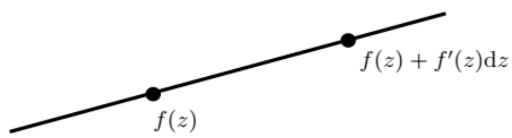
Define

$$\beta_j\pi = \pi - \alpha_j\pi,$$

so that $\beta_j\pi$ is the exterior angle ($\beta_j < 0$ for reentrant corners). Then

$$\sum_{j=1}^{n} \beta_j = 2, \qquad -2 \leq \beta_j \leq 2.$$

Now we map $y > 0$ onto D with the real axis mapping to $\mathrm{d}D$ and $x_1, x_2, \ldots, x_n$ mapping to the vertices $\zeta_1, \zeta_2, \ldots, \zeta_n$ by $\zeta = f(z)$. The tangent to $\mathrm{d}D$ has direction $\arg f'(z)$



(as $dz = dx$ is real on $\mathrm{d}D$) and this is constant on each side of $\mathrm{d}D$. At x_j, the preimage of vertex j, we have

$$\left[\arg f'(x)\right]_{x_j^-}^{x_j^+} = \beta_j\pi.$$

If we can find a function $f_j(z)$ such that

$$\arg f_j'(x) = \begin{cases} 0 & x > x_j, \\ -\beta_j\pi & x < x_j, \end{cases}$$

with $f_j'(z) \neq 0$ for $z \neq x_j$, then we can try

$$f'(z) = C\prod_{j=1}^{n} f_j'(z),$$

because then

$$\arg f'(z) = \arg C + \sum_j \arg f_j'(z)$$

has exactly the right properties. Just such a function is

$$f_j(z) = (z - x_j)^{-\beta_j},$$

and so a map from the upper-half plane (UHP) to D is $\zeta = f(z)$ where

$$\frac{\mathrm{d}f}{\mathrm{d}z} = C \prod_{j=1}^{n} (z - x_j)^{-\beta_j}.$$

Hence

$$\zeta = A + C \int^{z} \prod_{j=1}^{n} (t - x_j)^{-\beta_j}\, \mathrm{d}t, \tag{2}$$

where A and C fix the location and rotation/scaling of the polygon.

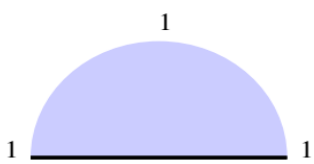
Notes

1 Can show (2) is a one-to-one map from $\{Imz > 0\}$ to D.

2 We are allowed by Riemann to fix the pre-images of 3 boundary points—that is, 3 of the x_j. Any more have to be found as part of the solution (by solving $f(x_j) = \zeta_j$). The best choice depends on the problem (*e.g.* using symmetry). Sometimes we take $x_n = \infty$ and then

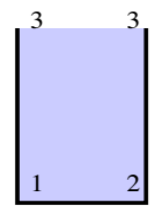
$$f(z) = A + C \int^{z} \prod_{j=1}^{n-1} (t - x_j)^{-\beta_j}\, \mathrm{d}t.$$

3 The definition of a polygon is elastic—it includes those with vertices at ∞ and those with interior angles of 2π.

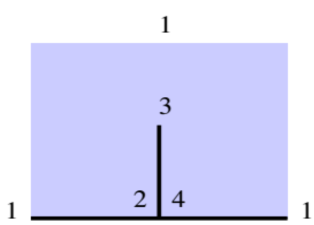


(a) $n = 1$, $\alpha_1 = -1$, $\beta_1 = 2$

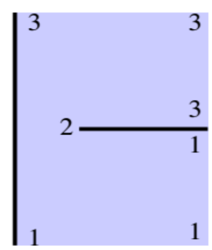
(b) $n = 2$, $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$



(c) $n = 3$, $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 0$, $\beta_1 = \beta_2 = 1/2$, $\beta_3 = 1$



(d) $n = 4$, $\alpha_1 = -1$, $\alpha_2 = \alpha_4 = 1/2$, $\alpha_3 = 2$, $\beta_1 = 2$, $\beta_2 = \beta_4 = 1/2$, $\beta_3 = -1$



(e) $n = 3$, $\alpha_1 = \alpha_3 = -1/2$, $\alpha_2 = 2$, $\beta_1 = \beta_3 = 3/2$, $\beta_2 = -1$

Most tractable examples are degenerate (*e.g.* they have a vertex at ∞) and use symmetry to simplify the integration.

Example: Map a half-plane to a strip with the vertices corresponding to $z = 0$ and $z = \infty$.

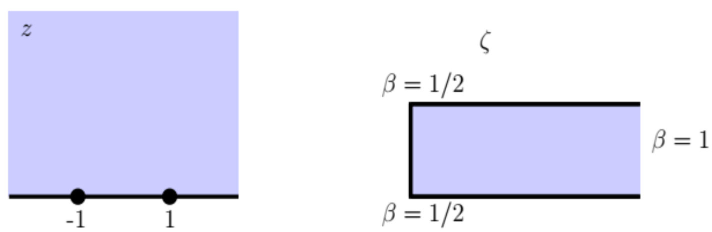
Solution Here ζ_1 and ζ_2 are both at ∞, with $\beta_1 = \beta_2 = 1$. Thus

$$\zeta = A + C\int^z \frac{\mathrm{d}t}{t} = A + C\log z.$$

If we want to map $z = x_1$ and $z = x_2$ to the ends of the strip we have

$$\zeta = A + C\int^z \frac{\mathrm{d}t}{(t-x_1)(t-x_2)} = A + \tilde{C}\log\left(\frac{z-x_1}{z-x_2}\right).$$

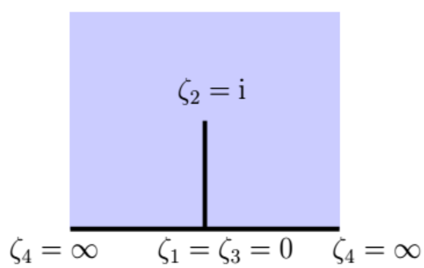
Example: Map a half-plane to a half-strip.



Solution Here $n = 3$, $\beta_1 = \beta_2 = 1/2$, $\beta_3 = 1$. It is convenient to take $x_1 = -1$, $x_2 = 1$, $x_3 = \infty$, to give

$$\zeta = A + C\int^z \frac{\mathrm{d}t}{\sqrt{t^2-1}} = A + C\cosh^{-1} z.$$

Example: Map UHP to the slit domain shown.



Solution Here take $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = \infty$. We have $\beta_1 = 1/2$, $\beta_2 = -1$, $\beta_3 = 1/2$, $\beta_4 = 2$. Thus

$$\zeta = A + C\int^z \frac{t}{\sqrt{t^2-1}}\,\mathrm{d}t = A + C\sqrt{z^2-1}.$$

$$\zeta_1 = \zeta_3 = 0 \quad\Rightarrow\quad \zeta = 0 \text{ when } z = \pm 1 \quad\Rightarrow\quad A = 0.$$

$$\zeta_2 = \mathrm{i} \quad\Rightarrow\quad \zeta = \mathrm{i} \text{ when } z = 0 \quad\Rightarrow\quad C = 1.$$

Thus

$$\zeta = \sqrt{z^2-1}.$$

Although this example has 4 vertices, symmetry gives an exact solution. In general, if the image is a quadrilateral we can only fix 3 vertices.