Lecture 8: Flow in porous media

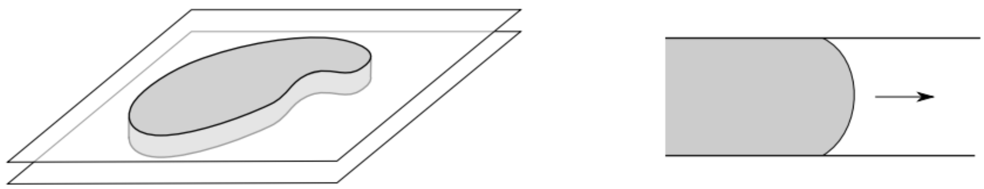
The nineteenth-century French hydraulic engineer Darcy formulated his law for flow of a viscous liquid through a porous matrix

$$\mathbf{u} = -k\nabla\left(p + \rho g Z\right) \qquad (Z \text{ vertically upwards}).$$

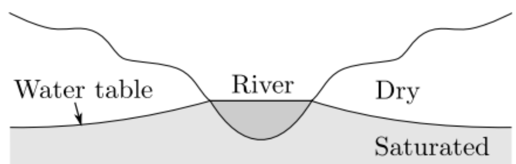
If the fluid is incompressible and the permeability k is constant then

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{so} \qquad \boxed{\nabla^2 p = 0}$$

Hence we can calculate flow with impermeable boundaries (e.g. goundwater, oil, ...) as for inviscid fluid flow. The same equations also arise in Hele-Shaw flow, in which a viscous fluid is forced between two parallel plates:

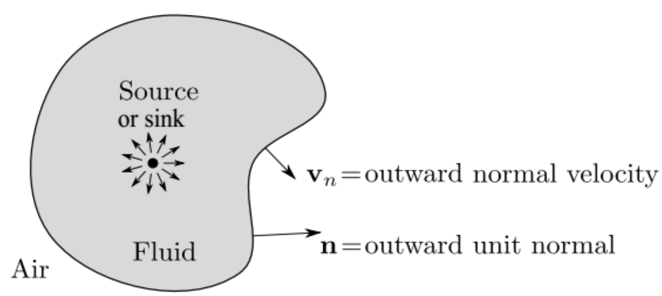


It is also possible to use hodograph methods to calculate steady flow with free surfaces, *i.e.* interfaces between saturated and unsaturated regions. For example



Even better, we can calculate the evolution of *unsteady* interfaces, provided we neglect gravity (*i.e.* we model 2D horizontal flow).

A canonical problem is injection or suction from a point source/sink into a porous medium (or Hele-Shaw cell)



Taking $k = 1$ (via nondimensionalisation) we have $\mathbf{u} = (u, v) = -\nabla p = \nabla\phi$ in the fluid, where $\phi = -p$ is the velocity potential. On the (moving) interface we need *two* conditions. One is

$$p = \text{constant} = 0 \qquad \text{(w.l.o.g.)},$$

and conservation of mass says that particles on the surface stay there, *i.e.*

$$\frac{\partial \phi}{\partial n} = -\frac{\partial p}{\partial n} = v_n,$$

where v_n is the normal velocity of the interface. A convenient way to write this is to note that since $p(x, y, t) = 0$ on the interface,

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = \frac{\partial p}{\partial t} - \nabla p \cdot \nabla p = 0,$$

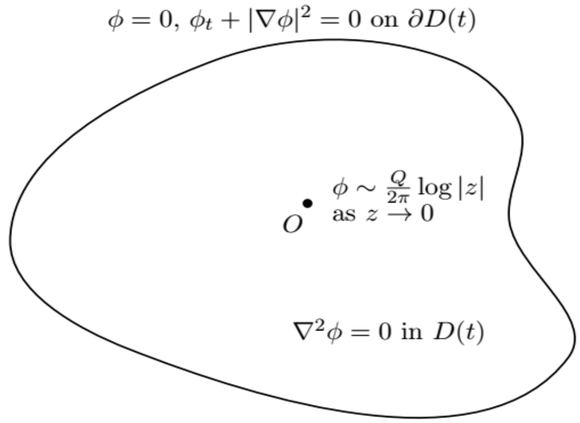
there, so the free boundary conditions are

$$\phi = 0, \qquad \frac{\partial \phi}{\partial t} + |\nabla \phi|^2 = 0.$$

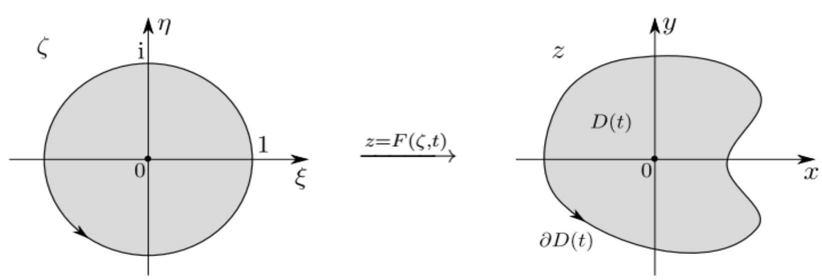
Lastly, near the source/sink at $z = 0$, we have

$$\phi \sim \frac{Q}{2\pi} \log |z|,$$

where $Q > 0$ is a source and $Q < 0$ is a sink.



As ϕ is a velocity potential we have a complex potential $w = \phi + \mathrm{i}\psi$. Now map D onto the unit disc $|\zeta| < 1$ with the sink mapped to $\zeta = 0$. It is convenient to write $z = F(\zeta, t)$ so that $F(0, t) = 0$ (& $F'(0)$ real and positive—Riemann).



We also write $W(\zeta, t) = w(z(\zeta, t), t)$. In the ζ-plane $W(\zeta, t)$ is a holomorphic function whose real part vanishes on $|\zeta| = 1$ (since $\phi = 0$ on the free boundary) and with a logarithmic singularity at $\zeta = 0$. Thus

$$W(\zeta, t) = \frac{Q}{2\pi} \log \zeta.$$

(branch cut OK because streamfunction increases by Q on circuit of O). The boundary condition

$$\frac{\partial \phi}{\partial t} + |\nabla \phi|^2 = 0$$

is equivalent to

$$Re \left(\frac{\partial W}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial W}{\partial t} \right) + \left| \frac{\partial W}{\partial \zeta} \right|^2 \Big/ \left| \frac{\partial z}{\partial \zeta} \right|^2 = 0.$$

Now use

$$W = \frac{Q}{2\pi} \log \zeta \qquad \left(\text{so } \frac{\partial W}{\partial t} = 0 \right)$$

and $z = F(\zeta, t) = F(\zeta(z,t), t)$, so that $0 = \dfrac{\partial F}{\partial \zeta}\dfrac{\partial \zeta}{\partial t} + \dfrac{\partial F}{\partial t}$ (on differentiating wrt t), to give

$$Re\left(-\frac{Q}{2\pi\zeta}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial \zeta}\right) + \frac{Q^2}{4\pi^2|\zeta|^2}\Big/\left|\frac{\partial F}{\partial \zeta}\right|^2 = 0.$$

On the free boundary $|\zeta| = 1$ this tidies up to

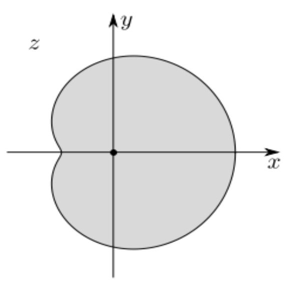
$$Re\left(\zeta\frac{\partial F}{\partial \zeta}\overline{\frac{\partial F}{\partial t}}\right) = \frac{Q}{2\pi}. \tag{10}$$

Given the initial shape $D(0)$ (*i.e.* $F(\zeta, 0)$) we can perhaps calculate the evolution. The miraculous result is the following:

Theorem If $F(\zeta, 0)$ is a polynomial of degree n, conformal and 1-1 for $|\zeta| < 1$, then for $t > 0$ $F(\zeta, t)$ is still a polynomial of degree n; moreover the (time-dependent) coefficients of this polynomial satisfy a set of $n+1$ o.d.e.s.

Example: Suppose $F(\zeta, 0) = a_{10}\zeta + a_{20}\zeta^2$, where a_{10}, a_{20} are real and positive, and $a_{10} > 2a_{20}$ (so that $F' \neq 0$). Find the evolution under a source/sink at the origin.

Solution The initial shape is a limaçon.



$$\left.\begin{aligned} x &= a_{10}\cos\theta + a_{20}\cos 2\theta, \\ y &= a_{10}\sin\theta + a_{20}\sin 2\theta, \end{aligned}\right\} \quad \zeta = e^{i\theta}.$$

We try $F(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$, a_1, a_2 real. Then

$$\frac{\partial F}{\partial t} = \dot{a}_1\zeta + \dot{a}_2(t)\zeta^2, \qquad \frac{\partial F}{\partial \zeta} = a_1 + 2a_2\zeta,$$

and the evolution equation (10) is

$$Re\left(\zeta(a_1 + 2a_2\zeta)(\dot{a}_1\overline{\zeta} + \dot{a}_2(t)\overline{\zeta}^2)\right) = \frac{Q}{2\pi} \quad \text{on } |\zeta| = 1.$$

Since $\zeta\overline{\zeta} = 1$ this is

$$Re\left(a_1\dot{a}_1 + 2a_2\dot{a}_2 + 2a_2\dot{a}_1\zeta + a_1\dot{a}_2\overline{\zeta}\right) = \frac{Q}{2\pi}.$$

which gives

$$a_1\dot{a}_1 + 2a_2\dot{a}_2 = \frac{Q}{2\pi}, \qquad 2a_2\dot{a}_1 + a_1\dot{a}_2 = 0.$$

The first equation is an exact differential which we can integrate directly to give

$$\frac{1}{2}\left(a_1^2 + 2a_2^2\right) = \frac{Qt}{2\pi} + \frac{1}{2}\left(a_{10}^2 + 2a_{20}^2\right)$$

(this is the area relation). Multiplying the second by a_1 turns that into an exact differential, which we can integrate to give

$$a_1^2 a_2 = a_{10}^2 a_{20}.$$

Lecture 8: Flow in porous media

If $Q > 0$ (injection) a_1 increases, a_2 decreases and $D(t)$ approaches a large circle. We can see that a_1 increases by observing that on eliminating a_2

$$a_1 \dot{a}_1 \left(1 - \frac{4a_2^2}{a_1^2}\right) = \frac{Q}{2\pi}.$$

Since $a_1 > 2a_2$ initially, $\dot{a}_1 > 0$ initially. But if a_1 increases then a_2 decreases so that $a_1 > 2a_2$ always. If $Q < 0$ (suction) however, a_1 decreases and a_2 increases. Soon we get to a time when $a_1 = 2a_2$ and the map has a *cusp* on the boundary (which is then a cardioid). At this time the solution ceases to exist.