Lecture 9: Plemelj formulae

Recall that if w is holomorphic inside and on the closed contour Γ and z is a point inside Γ, then Cauchy's integral formula states that

$$w(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{w(\zeta)\,\mathrm{d}\zeta}{\zeta - z}.$$

This relates w inside the contour to w on the contour.

Let us consider more generally the Cauchy type integral

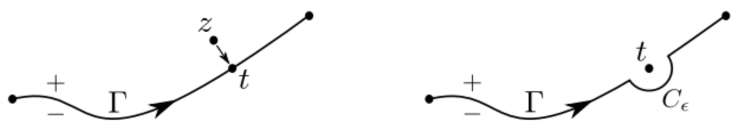
$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

where f is a given function on the contour Γ, which may now be closed or open.

If Γ is open, it is convenient in the subsequent analysis to adopt the convention that it does not contain its endpoints, a, $b \in \mathbb{C}$ say. Thus, an open contour may be parametrized by $\Gamma = \{\gamma(t) \in \mathbb{C} : t_0 < t < t_1\}$, where $a = \gamma(t_0) \neq \gamma(t_1) = b$ and $t_0 < t_1$ are real constants. We define $\overline{\Gamma} = \{\gamma(t) \in \mathbb{C} : t_0 \leq t \leq t_1\}$ to be the (topological) closure of Γ, *i.e.* $\overline{\Gamma}$ is the union of Γ and its endpoints. If Γ a closed contour, it may be parametrized by $\Gamma = \{\gamma(t) \in \mathbb{C} : t_0 \leq t < t_1\}$, where $a = \gamma(t_0) = \gamma(t_1) = b$, and the closure $\overline{\Gamma} = \Gamma$ because Γ is (topologically) closed.

If f is sufficiently smooth (*e.g.* continuous) on $\overline{\Gamma}$, then w is holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$.

Suppose t is any point at which Γ is smooth and that f is holomorphic in a neighbourhood of t and continuous on Γ. Let us label the left-hand side of Γ (as Γ is traversed in the direction of integration) as "+", and the right-hand side as "−". Let z approach $t \in \Gamma$ from the positive side as illustrated in figure 13(a).



(a) Original contour (b) Deformed contour

Figure 13: Deformed integration contour for $w_+(z)$.

We deform Γ near t by replacing $\gamma_\epsilon = \Gamma \cap D(t, \epsilon) \subset \Gamma$ with a small semi-circle C_ϵ as illustrated in figure 13(b), where ϵ is sufficiently small that f is holomorphic in the disc $D(t, 2\epsilon) = \{z : |z - t| < 2\epsilon\}$ say. By the deformation theorem,

$$w_+(t) = \lim_{z \to t} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\backslash\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\backslash\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta.$$

As $\varepsilon \to 0$, the semi-circle gives a residue contribution

$$\frac{1}{2} \times 2\pi \mathrm{i} \times \frac{f(t)}{2\pi \mathrm{i}} = \frac{1}{2} f(t);$$

note that only half the residue is included because we are only integrating over a semi-circle. Hence,

$$w_+(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma\backslash\gamma_\epsilon} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \;⨍_\Gamma \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta + \frac{1}{2} f(t), \tag{11}$$

where we define the principal value integral as

$$⨍_\Gamma = \lim_{\epsilon \to 0} \int_{\Gamma\backslash\gamma_\epsilon};$$

note that this always exists because the log singularities from the endpoints cancel because f is continuous on Γ.

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(a) Original contour (b) Deformed contour

Figure 14: Deformed integration contour for $w_-(z)$.

If we let $z \to \zeta \in \Gamma$ from the minus side as illustrated in figure 14(a), then we must deform Γ near $\zeta = t$ by replacing $\gamma_\epsilon \subset \Gamma$ with a small semi-circle C'_ϵ as illustrated in figure 14(b).

Again by the deformation theorem

$$w_-(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma \setminus \gamma_\epsilon} + \int_{C'_\epsilon} \right) \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \,⨍_\Gamma \frac{f(\zeta)}{\zeta - t} \mathrm{d}\zeta - \frac{1}{2} f(t). \tag{12}$$

In this case we are integrating in the opposite direction around the semi-circle, so that the residue contribution is $-f(t)/2$. Equations (11) and (12) are known as the **Plemelj formulae**. The conditions on f can be weakened (see *e.g.* Ablowitz & Fokas) and the formulae generalized to corners. We shall use them to construct solutions to Laplace's equation with a jump in value across Γ (*e.g.* aerofoil and fracture).

Quotable results (from Ablowitz & Fokas)

(1) If f is continuous on $\overline{\Gamma}$, then $w(z)$ is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ and $w(z) = \mathrm{O}(1/z)$ as $|z| \to \infty$.

(2) Behaviour at an endpoint t_e of Γ:

- (a) If $f(t) \to 0$ as $t \to t_e$, then $w(z) = \mathrm{O}(1)$ as $z \to t_e$ with $z \in \mathbb{C} \setminus \overline{\Gamma}$;
- (b) If $f(t) = \mathrm{O}(1)$ as $t \to t_e$, then $w(z) = \mathrm{O}(\log(z - t_e))$ as $z \to t_e$ with $z \in \mathbb{C} \setminus \overline{\Gamma}$;
- (c) If $f(t) = \mathrm{O}((t - t_e)^{-\alpha})$ as $t \to t_e$ for some $\alpha \in (0, 1)$, then $w(z) = \mathrm{O}((z - t_e)^{-\alpha})$ as $z \to t_e$ with $z \in \mathbb{C} \setminus \overline{\Gamma}$;

Applications of the Plemelj formulae

The problem of determining $w(z)$ in terms of its values on a curve Γ is ill-posed. The solution may not exist or may not be unique or it may not depend continuously on the boundary values.

Example: If $w(z)$ is holomorphic in $y > 0$ and

$$w(x) = \frac{\mathrm{i}\epsilon}{\delta} \tan^{-1}\left(\frac{x}{\delta}\right) \quad \text{for} \quad -\infty < x < \infty,$$

then

$$w(z) = \frac{\epsilon}{2\delta} \log\left(\frac{z + \mathrm{i}\delta}{z - \mathrm{i}\delta}\right) \to \infty \quad \text{as} \quad z \to \mathrm{i}\delta,$$

no matter how small ϵ/δ. Thus, despite the solution being arbitrarily small on the real axis, we only need to move δ away from the real axis for it to be infinite.

Hence, we will only consider well-posed problems in which either $\Re(w)$ or $Im(w)$ or some linear combination of w_+ and w_- is prescribed on Γ.

Problem 1: Find a function $w(z)$ holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ such that $w_+ - w_- = G$ on Γ, where G is continuous on $\overline{\Gamma}$.

Solution: Seek a solution for w of the form

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}.$$

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If we subtract the (12) from (11), we find $f = w_+ - w_- = G$ on Γ. Hence, a solution is given by

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(\zeta)\,\mathrm{d}\zeta}{\zeta - z} + h(z),$$

where $h(z)$ is an arbitrary function of z that is holomorphic on $\mathbb{C}\backslash\{a, b\}$.

Important remarks

- h is trivially a solution of the homogeneous problem in which $G = 0$.
- To pin down h, it is necessary to prescribe the behaviour of w at a, b and ∞ (via extra physics), for example, suppose

  (I) w is finite or has a logarithmic singularity at each of the endpoints of Γ;

  (II) there exists $n \in \mathbb{N}_0$ such that $w(z) = \mathrm{O}(z^n)$ as $|z| \to \infty$.

  Then, (I), the quotable results (2a,b) and Laurent's theorem imply that h can only have removable singularities at the endpoints of Γ, so that h is in fact entire. Hence, by (II) and the corollary to Liouville's theorem, $h(z) = p_n(z)$, an arbitrary polynomial of degree n.