Lecture 13: Use of multifunctions in Fourier transforms

Use of multifunctions in Fourier transforms

Example: A technical difficulty can arise in the theory of wave propagation. Looking for time-harmonic solutions of the wave equation

$$\nabla^2 \psi = \frac{1}{c^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}$$

in the form $\psi(x,y,t) = \phi(x,y)e^{-\mathrm{i}ct}$ leads to Helmholtz equation

$$\nabla^2\phi + \phi = 0.$$

Suppose that this holds in $y > 0$ with $\phi = f(x)$ on $y = 0$ and $\phi \to 0$ as $y \to \infty$, where $f(x)$ is piecewise smooth and decays sufficiently rapidly as $x \to \pm\infty$ that $\bar{f}(k)$ exists and is holomorphic in the strip $|Im(k)| < \gamma$ for some $\gamma > 0$. Taking the Fourier transform gives

$$\frac{\partial^2\overline{\phi}}{\partial y^2} - k^2\overline{\phi} + \overline{\phi} = 0.$$

Thus

$$\overline{\phi} = A(k)e^{-(k^2-1)^{1/2}\,y}.$$

where we require that $\Re(k^2-1)^{1/2} > 0$ on the inversion contour, so that $\overline{\phi} \to 0$ as $y \to \infty$. The boundary condition gives $A = \overline{f}$, so that, applying the inversion theorem,

$$\phi = \frac{1}{2\pi}\int_{\Gamma_1} \overline{f}(k)e^{-(k^2-1)^{1/2}\,y - \mathrm{i}kx}\,\mathrm{d}k,$$

where we require that $\Re(k^2-1)^{1/2} > 0$ on the inversion contour Γ_1.

Now $(k^2-1)^{1/2}$ has branch points at $k = \pm 1$. Suppose we take the + branch

$$(k^2-1)^{1/2} = (k-1)^{1/2}(k+1)^{1/2} = r_1^{1/2}r_2^{1/2}e^{i(\theta_1+\theta_2)/2},$$

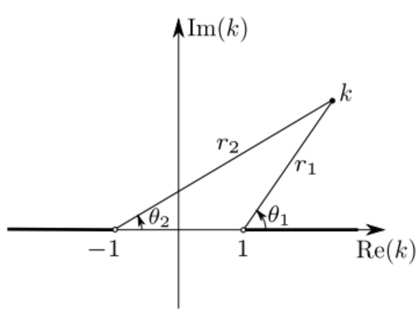
where $r_{1,2}$ and $\theta_{1,2}$ are defined in figure (a) below, with

$$0 < \theta_1 < 2\pi, \qquad -\pi < \theta_2 < \pi,$$

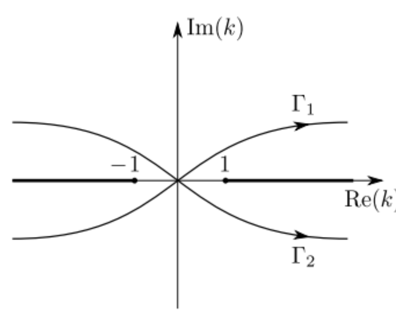
so that the branch cut is along the real axis from $-\infty$ to -1 and from 1 to ∞. Then, to ensure positivity of the real part of $(k^2-1)^{1/2}$, we require

$$-\pi < \theta_1 + \theta_2 < \pi.$$

This means that, with our choice of the + branch, the first and third quadrants are allowed, so a suitable inversion contour is Γ_1 in figure (b) below.



(a) Definition of $r_{1,2}$ and $\theta_{1,2}$



(b) Possible inversion contours

On the other hand, if we use the − branch,

$$(k^2-1)^{1/2} = -r_1^{1/2} r_2^{1/2} e^{i(\theta_1+\theta_2)/2},$$

with the same branch cut and domains of θ_1 and θ_2 as above, then another independent solution is

$$\phi = \frac{1}{2\pi}\int_{\Gamma_2} \overline{f}(k) e^{-(k^2-1)^{1/2}y - \mathrm{i}kx}\,\mathrm{d}k,$$

where we require that $\Re(k^2-1)^{1/2} > 0$ on the inversion contour Γ_2. In this case we require $\pi < \theta_1 + \theta_2 < 2\pi$, so that the second and fourth quadrants are allowed and a suitable inversion contour is Γ_2 in figure (b) above.

The problem as posed does not have a unique solution. To ensure uniqueness we need to impose an extra condition, *e.g.* there are no incoming waves as $r \to \infty$ only if

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} - \frac{\mathrm{i}\phi}{r} = o\left(\frac{1}{r}\right) \qquad \text{as } r^2 = x^2 + y^2 \to \infty,$$

which is known as the Sommerfeld radiation condition. This distinguishes Γ_1 from Γ_2.

Integral solutions of differential equations

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy.$$

Suppose we represent the solution as an integral

$$y(x) = \int_\Gamma g(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta \tag{22}$$

around some contour Γ. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy \quad\Leftrightarrow\quad \int_\Gamma \zeta g(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta = \int_\Gamma x g(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\zeta \tag{23}$$

$$= \int_\Gamma \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(g(\zeta)\mathrm{e}^{x\zeta}\right) - g'(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\eta \tag{24}$$

$$= \left[g(\zeta)\mathrm{e}^{x\zeta}\right]_\Gamma - \int_\Gamma g'(\zeta)\mathrm{e}^{x\zeta}\,\mathrm{d}\eta. \tag{25}$$

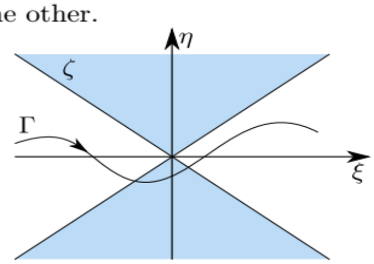
Thus we will have a solution to the differential equation only if

$$g' = -\zeta g$$

and the change in $g\mathrm{e}^{x\zeta}$ around Γ is zero. This gives $g = C\mathrm{e}^{-\zeta^2/2}$, and

$$y(x) = C\int_\Gamma \mathrm{e}^{x\zeta-\zeta^2/2}\,\mathrm{d}\zeta.$$

Now $g \to 0$ as $|\zeta| \to \infty$ providing $\Re(\zeta^2) > 0$, *i.e.* $-\pi/4 < \arg(\zeta) < \pi/4$ or $3\pi/4 < \arg(\zeta) < 5\pi/4$. Thus the contour Γ must start and end in one of these "valleys". If Γ starts and ends in the same valley the integral evaluates to zero (the contour can be deformed to infinity). This gives only one independent solution: a contour which starts in one valley and ends in the other.



Of course, for this simple example we can sinply take Γ along the real axis to obtain

$$y(x) = C\int_{-\infty}^{\infty} \mathrm{e}^{x\xi-\zeta^2/2}\,\mathrm{d}\xi = Ce^{x^2/2}\int_{-\infty}^{\infty} \mathrm{e}^{-(x-\xi)^2/2}\,\mathrm{d}\xi = C\sqrt{2\pi}\,e^{x^2/2},$$

as expected. This method also works for certain higher order ODEs such as

$$y'' + xy = 0 \quad \text{(Airy's eqn)}, \qquad xy'' + y' + xy = 0 \quad \text{(Bessel's eqn)}$$

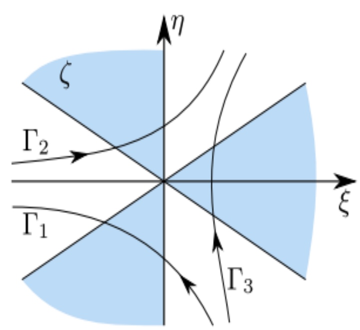
which lead to first-order ODEs for $g(\zeta)$. To solve the Airy equation, for example, we write the solutions for $y(x)$ as in (22). Similar to before we find

$$\left[g(\zeta)\mathrm{e}^{x\zeta}\right]_\Gamma + \int_\Gamma \left(\zeta^2 g(\zeta) - g'(\zeta)\right)\mathrm{e}^{x\zeta}\,\mathrm{d}\eta = 0.$$

Hence, the ODE is satisfied only if $\zeta^2 g(\zeta) = g'(\zeta)$ and $\left[g(\zeta)\mathrm{e}^{x\zeta}\right]_\Gamma$. Thus, $g(\zeta) = Ce^{\zeta^3/3}$ ($C \in \mathbb{C}$) and

$$y(x) = C\int_\Gamma \mathrm{e}^{x\zeta+\zeta^3/3}\,\mathrm{d}\zeta.$$

For fixed x we have $e^{x\zeta+\zeta^3/3} \to 0$ iff $\zeta \to \infty$ with $\Re(\zeta^3) < 0$, *i.e.* $\pi/6 < \arg\zeta < \pi/2$, $5\pi/6 < \arg\zeta < 7\pi/6$, or $-\pi/2 < \arg\zeta < -\pi/6$. Therefore, Γ should start and end in one of these sectors. For the integral in (22) to be non-zero, Γ should begin and end in a different sector which gives us three possibilities for Γ leading to three different solutions:



However, we can write one of these solutions as the sum of the other two by deforming the contours of integration which means that we only have two independent solutions (which is what we would expect since we are solving a second-order linear ODE).

This method is equivalent to formally taking a Fourier transform of the equation, and then choosing an inversion contour so that the resulting solution exists. Note that if the coefficients in the equation were x^2 rather than x then the equation for g would be second-order rather than first-order (and may therefore be as hard to solve for g as the original equation is for y).