

LECTURE 13: SPECTRAL COLLOCATION

- Idea: Rather than use low-degree local interpolants, use the above to construct global differentiation matrices for high degree global interpolants.
- Equispaced points will be bad, but Chebyshev will be great!
- Interpolate data $\{x_k, v_k\}_{k=0}^n$ to evaluate derivative in $[x_0, x_n]$.
- Global interpolants lead to geometric convergence, but dense matrices.
- demo: `diff_sine.m`, `diff_sine2.m`

BOUNDARY VALUE PROBLEMS (BVPs)

BVPs

- Question: If D approximates a derivative, what about D^{-1} ?
- Answer: Approximates antiderivative / indefinite integral.

$$Lu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1.7071 & 0.7071 & 1.4142 & -0.7071 & 0.2929 \\ 0.5000 & -1.4142 & 0 & 1.4142 & -0.5000 \\ -0.2929 & 0.7071 & -1.4142 & -0.7071 & 1.7071 \\ 0.5000 & -1.1716 & 2.0000 & -6.8284 & 5.5000 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \approx \begin{pmatrix} 0 \\ u_1 \\ u_2' \\ u_3' \\ u_4' \end{pmatrix}$$

- Problem: D is rank deficient (it maps $\mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$).
- This is analogous to the unknown constant of integration.
- We can fix this by enforcing an additional constraint, e.g., that $u(-1) = 0$.
- L is now invertible, and given u' we can recover u .
- demo: `bvp0.m`

- More generally, we can use this approach to solve linear BVPs.
- Consider the simple oscillator

$$u'' + u = 0 \rightarrow D^2u + u = (D^2 + I)u = Lu = 0.$$

$$Lu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2.9142 & 1.5000 & 2.0000 & 0.5000 & 0.0858 \\ 0.2500 & 2.0000 & 1.0000 & 2.0000 & 0.2500 \\ 0.0858 & 0.5000 & 2.0000 & 1.5000 & 2.9142 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- 2nd-order ODE \rightarrow we need two boundary conditions to solve it uniquely.
- Let's choose $u(-1) = u(1) = 1$.
- demo: `bvp1.m`

- As before, consider the simple oscillator

$$u'' + u = 0 \rightarrow D^2u + u = (D^2 + I)u = Lu = 0.$$

but instead we wish to enforce

$$u(-1) = 1 \text{ and } u'(1) = 0.$$

- $g'(x_n) = \underline{d}_n g(\underline{x}) \rightarrow$ replace the last row of L with last row of D

$$Lu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2.9142 & 1.5000 & 2.0000 & 0.5000 & 0.0858 \\ 0.2500 & 2.0000 & 1.0000 & 2.0000 & 0.2500 \\ 0.0858 & 0.5000 & 2.0000 & 1.5000 & 2.9142 \\ 0.5000 & -1.1716 & 2.0000 & -6.8284 & 5.5000 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- demo: `bvp2.m`

- What about the BVP:

$$\begin{aligned} u'' + \sin(x)u &= 0 \\ u(-1) = 1, u'(1) &= 0. \end{aligned}$$

- Well,

$$\begin{aligned} u'' + \sin(x)u = 0 &\rightarrow D^2u + \sin(x) \cdot u = 0 \\ &\rightarrow (D^2 + \text{diag}(\sin(x)))u = Lu = 0 \end{aligned}$$

- The boundary conditions can be enforced as before.
- demo: `bvp3.m`

Nonlinear problems

- Nonlinear problems → Newton's method.
- Consider

$$F(u) = \frac{1}{20}u'' + \sin(u) = 0$$

$$u(-1) = 1, u(1) = 0.$$

Nonlinear problems

- Nonlinear problems → Newton's method.
- Consider

$$F(u) = \frac{1}{20}u'' + \sin(u) = 0$$

$$u(-1) = 1, u(1) = 0.$$

- For Newton's method we need a Gâteaux derivative.

$$J(u; \Delta u) = \frac{\partial F}{\partial u}(x, u; \Delta u) = \lim_{\epsilon \rightarrow 0} \frac{F(u + \epsilon \Delta u) - F(u)}{\epsilon}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{\frac{1}{20}D^2(u + \epsilon \Delta u) + \text{diag}(\sin(u + \epsilon \Delta u)) - (\frac{1}{20}D^2u + \text{diag}(\sin(u)))}{\epsilon}$$

$$= \left[\frac{1}{20}D^2 + \text{diag}(\cos(u)) \right] \Delta u$$

- demo: `bvp4.m`

More examples!

- My favourite linear ODE:

$$0.0005u'' + x(x^2 - \frac{1}{2})u' + 3(x^2 - \frac{1}{2})u = 0, u(-1) = 1, u(1) = 0.$$

- System of equations:

$$\begin{aligned} u'' - v &= 0 & u(-1) &= 1, & u'(1) &= 0 \\ u + v'' &= 0 & v'(-1) &= 0, & v(1) &= 0 \end{aligned}$$

- Nonlinear system of equations:

$$\begin{aligned} u'' - \sin(v) &= 0 & u(-1) &= 1, & u'(1) &= 0 \\ \cos(u) + v'' &= 0 & v'(-1) &= 0, & v(1) &= 0 \end{aligned}$$

- Eigenvalue problem: (Airy)

$$0.002u'' + ixu = \lambda u, \quad u(\pm 1) = 0.$$

Final remarks

- Finite differences → sparse matrices but slow convergence.
- Spectral methods → dense matrices but fast convergence.
- For complex domains: finite volumes, finite elements.
- (These methods also have their roots in polynomial interpolation!)
- Software (for 1D computations): Chebfun.
- Software (for 2D/3D computations): FEniCS.
- Often adaptive approaches are used (h and p refinement).
- BVPs vs IVPs - methods of solution are very different.