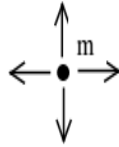


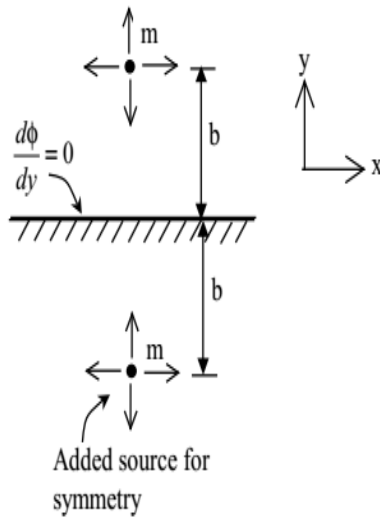
Lecture 11

3.11 - Method of Images

- Potential for single source: $\phi = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}$



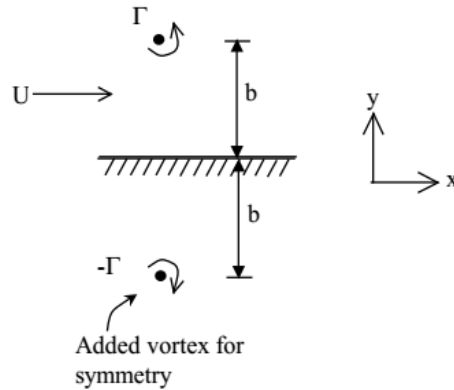
- Potential for source near a wall: $\phi = \frac{m}{2\pi} \left(\ln \sqrt{x^2 + (y - b)^2} + \ln \sqrt{x^2 + (y + b)^2} \right)$



Note: Be sure to verify that the boundary conditions are satisfied by symmetry or by calculus for $\phi(y) = \phi(-y)$.

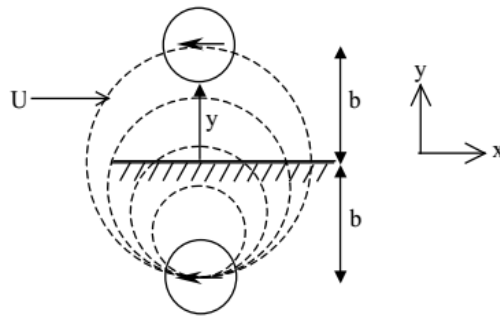
MARINE HYDRODYNAMICS II

- **Vortex near a wall (ground effect):** $\phi = Ux + \frac{\Gamma}{2\pi} \left(\tan^{-1}\left(\frac{y-b}{x}\right) - \tan^{-1}\left(\frac{y+b}{x}\right) \right)$



Verify that $\frac{d\phi}{dy} = 0$ on the wall $y = 0$.

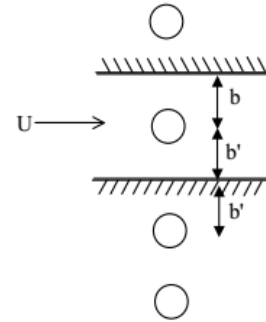
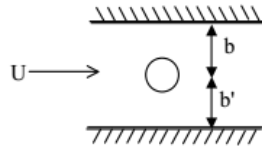
- **Circle of radius a near a wall:** $\phi \cong Ux \left(1 + \frac{a^2}{x^2 + (y-b)^2} + \frac{a^2}{x^2 + (y+b)^2} \right)$



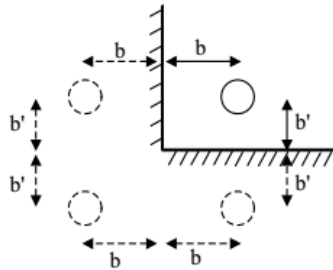
This solution satisfies the boundary condition on the wall ($\frac{\partial\phi}{\partial n} = 0$), and the degree it satisfies the boundary condition of no flux through the circle boundary increases as the ratio $b/a \gg 1$, i.e., the velocity due to the image dipole small on the real circle for $b \gg a$. For a 2D dipole, $\phi \sim \frac{1}{a}$, $\nabla\phi \sim \frac{1}{a^2}$.

- More than one wall:

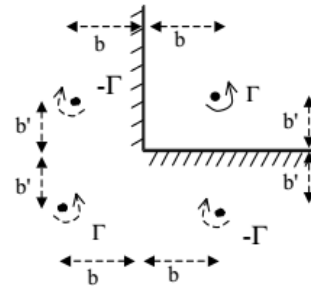
Example 1:



Example 2:

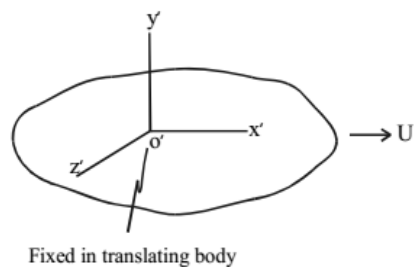
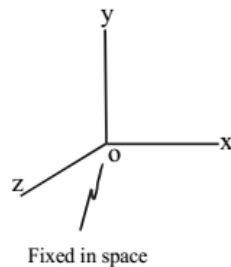


Example 3:


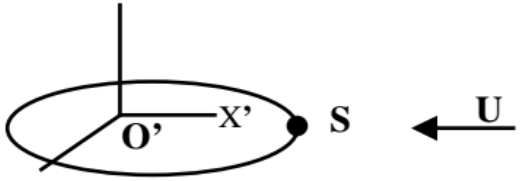


3.12 Forces on a body undergoing steady translation “D’Alembert’s paradox”

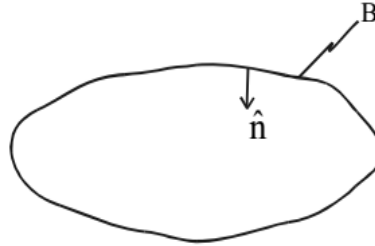
3.12.1 Fixed bodies & translating bodies - Galilean transformation.



$$\mathbf{x} = \mathbf{x}' + \mathbf{U}t$$

Reference system O: \vec{v}, ϕ, p	Reference system O': \vec{v}', ϕ', p'
	
$\nabla^2 \phi = 0$ $\vec{v} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n} = (U, 0, 0) \cdot (n_x, n_y, n_z)$ $= U n_x \text{ on Body}$ $\vec{v} \rightarrow 0 \text{ as } \vec{x} \rightarrow \infty$ $\phi \rightarrow 0 \text{ as } \vec{x} \rightarrow \infty$	$\nabla^2 \phi' = 0$ $\vec{v}' \cdot \hat{n}' = \frac{\partial \phi'}{\partial n} = 0$ $\vec{v}' \rightarrow (-U, 0, 0) \text{ as } \vec{x}' \rightarrow \infty$ $\phi' \rightarrow -Ux' \text{ as } \vec{x}' \rightarrow \infty$
<p>Galilean transform:</p> $\vec{v}(x, y, z, t) = \vec{v}'(x' = x - Ut, y, z, t) + (U, 0, 0)$ $\phi(x, y, z, t) = \phi'(x' = x - Ut, y, z, t) + Ux' \Rightarrow$ $-Ux' + \phi(x = x' + Ut, y, z, t) = \phi'(x', y, z, t)$	
<p>Pressure (no gravity)</p> $p_\infty = -\frac{1}{2}\rho v^2 + C_o = C_o = -\frac{1}{2}\rho v'^2 + C'_o = C'_o - \frac{1}{2}\rho U^2$ $\therefore C_o = C'_o - \frac{1}{2}\rho U^2$	
<p>In O: unsteady flow</p> $p_s = -\rho \frac{\partial \phi}{\partial t} - \frac{1}{2}\rho \underbrace{v^2}_{U^2} + C_o$ $\frac{\partial \phi}{\partial t} = \left(\underbrace{\frac{\partial}{\partial t}}_0 + \underbrace{\frac{\partial x'}{\partial t} \frac{\partial}{\partial x'}}_{-U} \right) (\phi' + Ux') = -U^2$ $\therefore p_s = \rho U^2 - \frac{1}{2}\rho U^2 + C_o = \frac{1}{2}\rho U^2 + C_o$ $p_s - p_\infty = \frac{1}{2}\rho U^2 \text{ stagnation pressure}$	<p>In O': steady flow</p> $p_s = -\rho \underbrace{\frac{\partial \phi'}{\partial t}}_0 - \frac{1}{2}\rho \underbrace{v'^2}_0 + C'_o = C'_o$ $p_s - p_\infty = \frac{1}{2}\rho U^2 \text{ stagnation pressure}$

3.12.2 Forces



Total fluid force for ideal flow (i.e., no shear stresses): $\vec{F} = \oiint_B p \hat{n} dS$

For potential flow, substitute for p from Bernoulli:

$$\vec{F} = \oiint_B -\rho \left(\underbrace{\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2}_{\text{hydrodynamic force}} + \underbrace{gy}_{\text{hydrostatic force}} + c(t) \right) \hat{n} dS$$

For the hydrostatic case ($\vec{v} \equiv \phi \equiv 0$):

$$\vec{F}_s = \oiint_B (-\rho g y \hat{n}) dS \stackrel{\substack{\uparrow \\ \text{Gauss} \\ \text{theorem}}}{=} (-) \underbrace{\iint_{v_B}}_{\substack{\uparrow \\ \text{outward} \\ \text{normal}}} \nabla (-\rho g y) dv = \underbrace{\rho g \nabla j}_{\substack{\uparrow \\ \text{Archimedes} \\ \text{principle}}} \text{ where } \nabla = \iiint_{v_B} dv$$

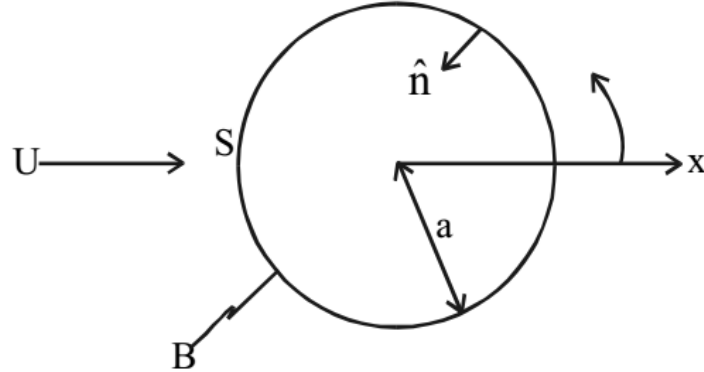
We evaluate **only** the hydrodynamic force:

$$\vec{F}_d = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \hat{n} dS$$

For steady motion ($\frac{\partial \phi}{\partial t} \equiv 0$):

$$\vec{F}_d = -\frac{1}{2} \rho \iint_B v^2 \hat{n} dS$$

3.12.3 **Example** Hydrodynamic force on 2D cylinder in a steady uniform stream.



$$\begin{aligned} \vec{F}_d &= \int_B \left(\frac{-\rho}{2}\right) |\nabla\phi|^2 \hat{n} dl = \int_0^{2\pi} \left(\frac{-\rho}{2}\right) |\nabla\phi|_{r=a}^2 \hat{n} a d\theta \\ F_x &= \vec{F} \cdot \hat{i} = \frac{-\rho a}{2} \int_0^{2\pi} d\theta |\nabla\phi|_{r=a}^2 \underbrace{\hat{n} \cdot \hat{i}}_{-\cos\theta} \\ &= \frac{\rho a}{2} \int_0^{2\pi} |\nabla\phi|_{r=a}^2 \cos\theta d\theta \end{aligned}$$

Velocity potential for flow past a 2D cylinder:

$$\phi = U r \cos\theta \left(1 + \frac{a^2}{r^2}\right)$$

Velocity vector on the 2D cylinder surface:

$$\nabla\phi|_{r=a} = (v_r|_{r=a}, v_\theta|_{r=a}) = \left(\underbrace{\frac{\partial\phi}{\partial r}}_0 \bigg|_{r=a}, \underbrace{\frac{1}{r} \frac{\partial\phi}{\partial\theta}}_{-2U \sin\theta} \bigg|_{r=a} \right)$$

Square of the velocity vector on the 2D cylinder surface:

$$|\nabla\phi|_{r=a}^2 = 4U^2 \sin^2\theta$$

Finally, the **hydrodynamic force** on the 2D cylinder is given by

$$F_x = \frac{\rho a}{2} \int_0^{2\pi} d\theta (4U^2 \sin^2 \theta \cos \theta) = \underbrace{\left(\frac{1}{2}\rho U^2\right)}_{p_s - p_\infty} \underbrace{(2a)}_{\substack{\text{diameter} \\ \text{or} \\ \text{projection}}} \underbrace{2 \int_0^{2\pi} d\theta \underbrace{\sin^2 \theta}_{\text{even}} \underbrace{\cos \theta}_{\substack{\text{odd} \\ \text{w.r.t. } \frac{\pi}{2}, \frac{3\pi}{2}}}}_{\equiv 0} = 0$$

Therefore, $F_x = 0 \Rightarrow$ no horizontal force (symmetry fore-aft of the streamlines). Similarly,

$$F_y = \left(\frac{1}{2}\rho U^2\right)(2a)2 \int_0^{2\pi} d\theta \sin^2 \theta \sin \theta = 0$$

In fact, *in general* we find that $\vec{F} \equiv 0$, on *any* 2D or 3D body.

D'Alembert's "paradox":

No hydrodynamic force* acts on a body moving with steady translational (no circulation) velocity in an infinite, inviscid, irrotational fluid.

* The moment as measured in a local frame is not necessarily zero.

3.13 Lift due to Circulation

3.13.1 **Example** Hydrodynamic force on a vortex in a uniform stream.

$$\phi = Ux + \frac{\Gamma}{2\pi}\theta = Ur \cos\theta + \frac{\Gamma}{2\pi}\theta$$



Consider a control surface in the form of a circle of radius r centered at the point vortex. Then according to Newton's law:

$$\begin{aligned} \Sigma \vec{F} &= \frac{d}{dt} \vec{\mathcal{L}}_{CV} \xrightarrow{\text{steady flow}} \\ (\vec{F}_V + \vec{F}_{CS}) + \vec{M}_{NET} &= 0 \Leftrightarrow \vec{F} \equiv -\vec{F}_V = \vec{F}_{CS} + \vec{M}_{NET} \end{aligned}$$

Where,

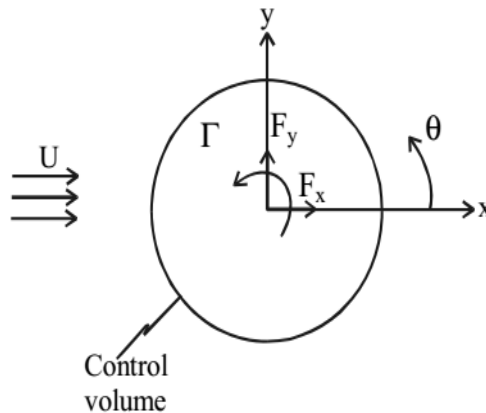
\vec{F} = Hydrodynamic force exerted on the vortex from the fluid.

$\vec{F}_V = -\vec{F}$ = Hydrodynamic force exerted on the fluid in the control volume from the vortex.

\vec{F}_{CS} = Surface force (i.e., pressure) on the fluid control surface.

\vec{M}_{NET} = Net linear momentum flux in the control volume through the control surface.

$\frac{d}{dt} \vec{\mathcal{L}}_{CV}$ = Rate of change of the total linear momentum in the control volume.



The hydrodynamic force on the vortex is $\vec{F} = \vec{F}_{CS} + \vec{M}_{IN}$

MARINE HYDRODYNAMICS II

- a. Net linear momentum flux in the control volume through the control surfaces, \vec{M}_{NET} . Recall that the control surface has the form of a circle of radius r centered at the point vortex.

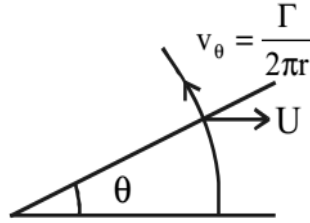
- a.1 The velocity components on the control surface are

$$u = U - \frac{\Gamma}{2\pi r} \sin \theta$$

$$v = \frac{\Gamma}{2\pi r} \cos \theta$$

The radial velocity on the control surface is therefore, given by

$$u_r = U \frac{\partial x}{\partial r} = U \cos \theta = \vec{V} \cdot \hat{n}$$



- a.2 The net horizontal and vertical momentum fluxes through the control surface are given by

$$(M_{NET})_x = -\rho \int_0^{2\pi} d\theta r u v_r = -\rho \int_0^{2\pi} d\theta r \left(U - \frac{\Gamma}{2\pi r} \sin \theta \right) U \cos \theta = 0$$

$$(M_{NET})_y = -\rho \int_0^{2\pi} d\theta r v v_r = -\rho \int_0^{2\pi} d\theta r \left(\frac{\Gamma}{2\pi r} \cos \theta \right) U \cos \theta$$

$$= -\frac{\rho U \Gamma}{2\pi} \int_0^{2\pi} \cos^2 \theta d\theta = -\frac{\rho U \Gamma}{2}$$

b. Pressure force on the control surface, \vec{F}_{CS} .

b.1 From Bernoulli, the pressure on the control surface is

$$p = -\frac{1}{2}\rho |\vec{v}|^2 + C$$

b.2 The velocity $|\vec{v}|^2$ on the control surface is given by

$$\begin{aligned} |\vec{v}|^2 &= u^2 + v^2 = \left(U - \frac{\Gamma}{2\pi r} \sin \theta \right)^2 + \left(\frac{\Gamma}{2\pi r} \cos \theta \right)^2 \\ &= U^2 - \frac{\Gamma}{\pi r} U \sin \theta + \left(\frac{\Gamma}{2\pi r} \right)^2 \end{aligned}$$

b.3 Integrate the pressure along the control surface to obtain \vec{F}_{CS}

$$\begin{aligned} (F_{CS})_x &= \int_0^{2\pi} d\theta r p (-\cos \theta) = 0 \\ (F_{CS})_y &= \int_0^{2\pi} d\theta r p (-\sin \theta) = \left(-\frac{\rho}{2}\right) \left(-\frac{\Gamma U}{\pi r}\right) (-r) \underbrace{\int_0^{2\pi} d\theta \sin^2 \theta}_{\pi} = -\frac{1}{2}\rho U \Gamma \end{aligned}$$

c. Finally, the force on the vortex \vec{F} is given by

$$\begin{aligned} F_x &= (F_{CS})_x + (M_x)_{IN} = 0 \\ F_y &= (F_{CS})_y + (M_y)_{IN} = -\rho U \Gamma \end{aligned}$$

i.e., the fluid exerts a downward force $F = -\rho U \Gamma$ on the vortex.

Kutta-Joukowski Law

$$\begin{aligned} 2D : F &= -\rho U \Gamma \\ 3D : \vec{F} &= \rho \vec{U} \times \vec{\Gamma} \end{aligned}$$

Generalized Kutta-Joukowski Law:

$$\vec{F} = \rho \vec{U} \times \left(\sum_{i=1}^n \vec{\Gamma}_i \right)$$

where \vec{F} is the total force on a system of n vortices in a free stream with speed \vec{U} .