

Introduction

Governing Equations so far:

Knowns	Equations	#	Unknowns	#
density $\rho(\vec{x}, t)$	Continuity (conservation of mass)	1	velocities $v_i(\vec{x}, t)$	3
body force F_i	Euler (conservation of momentum)	3	stresses $\tau_{ij}(\vec{x}, t)$	⊛6
		4		9

⊛ 3 of the 9 unknowns of the stress tensor are eliminated by symmetry

The number of unknowns (9) is $>$ than the number of equations (4), i.e. we don't have *closure*. We need *constitutive laws* to relate the kinematics v_i to the dynamics τ_{ij} .

1.9 Newtonian Fluids

1. Consider a fluid at rest ($v_i \equiv 0$). Then according to Pascal's Law:

$$\tau_{ij} = -p_s \delta_{ij} \text{ (Pascal's Law)}$$

$$\tau = \begin{bmatrix} -p_s & 0 & 0 \\ 0 & -p_s & 0 \\ 0 & 0 & -p_s \end{bmatrix}$$

where p_s is the hydrostatic pressure and δ_{ij} is the Kroenecker delta function, equal to 1 if $i = j$ and 0 if $i \neq j$.

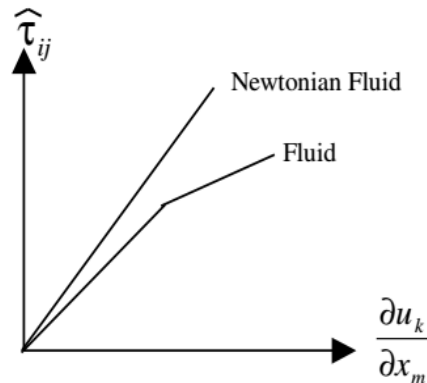
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2. Consider a fluid in motion. The fluid stress is defined as:

$$\tau_{ij} \equiv \underbrace{-p\delta_{ij}}_{\substack{\text{isotropic components} \\ \text{on diagonal}}} + \underbrace{\hat{\tau}_{ij}}_{\substack{\text{all non-isotropic components} \\ \text{both on/off diagonal}}}$$

where p is the thermodynamic pressure and $\hat{\tau}_{ij}$ are the dynamic stresses. It should be emphasized that $-p\delta_{ij}$ includes all the isotropic components of the stress tensor *on the diagonal*, while $\hat{\tau}_{ij}$ represents all the non-isotropic components, which may or may not be on the diagonal (shear *and* normal stresses). The dynamic stresses $\hat{\tau}_{ij}$ is related to the velocity gradients by *empirical* relations.

Experiments with a wide class of ‘*Newtonian*’ fluids showed that the dynamic stresses are proportional to the rate of strain.



$\hat{\tau}_{ij} \approx$ linear function of the (rate of strain \equiv velocity gradient)

$$\frac{\partial}{\partial t} \left(\frac{\partial X}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial X}{\partial t} \right) \quad \frac{\partial u_k}{\partial x_m}$$

u

i.e. $\hat{\tau}_{ij} \approx \underbrace{\alpha_{ijklm}}_{\substack{3^4=81 \\ \text{empirical coefficients} \\ \text{(constants for Newtonian fluids)}}} \frac{\partial u_k}{\partial x_m} \quad i, j, k, m = 1, 2, 3$

For isotropic fluids, this reduces to:

$$\hat{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \underbrace{\left(\frac{\partial u_l}{\partial x_l} \right)}_{\nabla \cdot \vec{v}},$$

The fluid properties in the previous relation are:

- μ - (coefficient of) dynamic viscosity.
- λ - bulk elasticity, ‘second’ coefficient of viscosity

For incompressible flow, we have shown that $\nabla \cdot \vec{v} = \frac{\partial u_i}{\partial x_i} = 0$.

Therefore, for an incompressible, isotropic, Newtonian fluid the dynamic or **viscous** stresses $\hat{\tau}_{ij}$ are expressed as:

$$\hat{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

1.9.1 Discussion on viscous stresses $\hat{\tau}_{ij}$

1. Verify that for $\vec{v} = 0$ we recover Pascal’s law.

Proof: $\vec{v} = 0 \Rightarrow \frac{\partial u_i}{\partial x_j} = 0 \Rightarrow \hat{\tau}_{ij} = 0 \Rightarrow \tau_{ij} = -p\delta_{ij} + 0 \Rightarrow$ hydrostatic conditions

2. Verify that $\hat{\tau}_{ij} = \hat{\tau}_{ji}$.

Proof: $\hat{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \hat{\tau}_{ji} \Rightarrow$ symmetry of stress tensor

3. When $\hat{\tau}_{ij}$, $i = j$ the viscous stress is a **normal stress** and is given by:

$$\hat{\tau}_{ii} = 2\mu \frac{\partial u_i}{\partial x_i}$$

The normal viscous stresses $\hat{\tau}_{ii}$ are the diagonal terms of the viscous stress tensor.

The $\hat{\tau}_{ii}$ in general are **not isotropic**.

4. When $\hat{\tau}_{ij}$, $i \neq j$ the viscous stress is a **shear stress** and is given by:

$$\hat{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The shear viscous stresses $\hat{\tau}_{ij}$, $i \neq j$ are the off diagonal terms of the viscous stress tensor.

5. A 2D viscous tensor has the form: $\mu \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} \end{bmatrix}$

6. Notation 1: The viscous stresses $\hat{\tau}_{ij}$ are often referred to (somewhat confusingly) as *shear* stresses, despite the fact that when $i = j$ the viscous stress is a normal stress.

7. Notation 2: Often $\tau_{ij} \leftrightarrow \hat{\tau}_{ij}$ and τ_{ij} is used to denote $\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$.

1.10 Navier-Stokes equations(for Incompressible, Newtonian Fluid)

Equations	#	Unknowns	#
Continuity	1	velocities $v_i(\vec{x}, t)$	3
Euler	3	stresses $\hat{\tau}_{ij}(\vec{x}, t)$	6
Newtonian fluid <small>symmetry</small>	6	pressure $p(\vec{x}, t)$	1
	10		10

closure

To form the Navier-Stokes equations for incompressible, Newtonian fluids, we first substitute the equation for the stress tensor for a Newtonian fluid, i.e.

$$\tau_{ij} = -p\delta_{ij} + \hat{\tau}_{ij} \quad \underset{\substack{= \\ \uparrow \\ \text{Newtonian Fluid}}}{=} \quad -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

into Euler's equation:

$$\begin{aligned} \rho \frac{Du_i}{Dt} &= F_i + \frac{\partial \tau_{ij}}{\partial x_j} \\ &= F_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ &= F_i - \frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \underbrace{\left(\frac{\partial u_j}{\partial x_j} \right)}_{\text{Continuity} = 0} \right) \end{aligned}$$

Therefore the Navier-Stokes equations for an incompressible, Newtonian fluid in cartesian coordinates are given as:

$$\begin{aligned} \frac{Du_i}{Dt} &= \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\rho} F_i && \text{Tensor form} \\ \frac{D\vec{v}}{Dt} &= \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{F} && \text{Vector form} \end{aligned}$$

where $\nu \equiv \frac{\mu}{\rho}$ denoted as the **kinematic viscosity** [L^2/T].

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- Unknowns and governing equations for incompressible, Newtonian fluids

Equations	#	Unknowns	#
Continuity	1	pressure $p(\vec{x}, t)$	1
Navier-Stokes	3	velocities $v_i(\vec{x}, t)$	3
	4		4

- Values of constants (density, dynamic and kinematic viscosity) used in 13.021

		water	air	units
density	ρ	10^3	1	[kg/m ³]
dynamic viscosity	μ	10^{-3}	10^{-2}	[kg/ms]
kinematic viscosity	ν	10^{-6}	10^{-5}	[m ² /s]

- Notation 1: The Continuity and the Navier-Stokes equations form the Governing Equations for incompressible, Newtonian fluids.

Continuity + Navier-Stokes = Governing Equations

Notation 2: Alternatively, we refer to each equation $\frac{Dv_i}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 v_i + \frac{1}{\rho} F_i$ as the i^{th} **Momentum** Equation. In this case, the Continuity and the Momentum equations form the Navier-Stokes System of Equations.

Continuity + Momentum Equations = Navier-Stokes System of Equations

Both notations are equivalent, and in this text it will be made clear from the context when the term Navier-Stokes refers to the Momentum Equations or to the System of Governing Equations.

1.11 Boundary Conditions

In the previous paragraphs we formulated the governing equations that describe the flow of an incompressible, Newtonian fluid. The governing equations (N-S) are a system of partial differential equations (PDE's). This 4×4 system of equations describes all the incompressible flows, from rain droplets to surface waves.

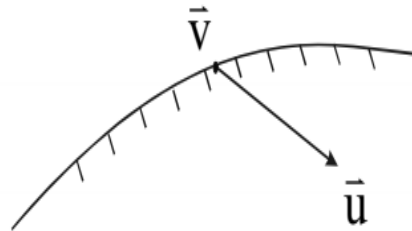
One of the reasons this system of equations provides such different solutions lies on the variety of the imposed **boundary conditions**. To complete the description of this problem it is imperative that we specify appropriate boundary conditions. For the N-S equations we need to specify 'Kinematic Boundary Conditions' and 'Dynamic Boundary Conditions'.

1. **Kinematic Boundary Conditions** specify the boundary kinematics (position, velocity, ...). On an *impermeable* solid boundary, velocity of the fluid = velocity of the body. i.e. velocity continuity.

$$\vec{v} = \vec{u} \quad \text{'no-slip' + 'no-flux' boundary condition}$$

where \vec{v} is the fluid velocity at the body and \vec{u} is the body surface velocity

- $\vec{v} \cdot \hat{n} = \vec{u} \cdot \hat{n}$ 'no flux' ← continuous flow
- $\vec{v} \cdot \hat{t} = \vec{u} \cdot \hat{t}$ 'no slip' ← finite shear stress



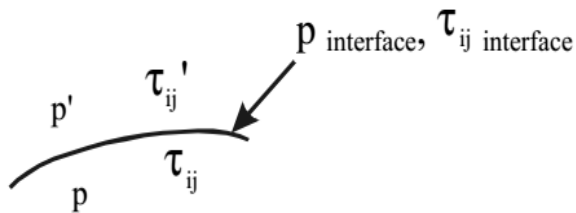
2. **Dynamic Boundary Conditions** specify the boundary dynamics (pressure, shear stress, ...).

Stress continuity:

$$p = p' + p_{\text{interface}}$$

$$\tau_{ij} = \tau'_{ij} + \tau_{ij, \text{interface}}$$

The most common example of interfacial stress is surface tension.



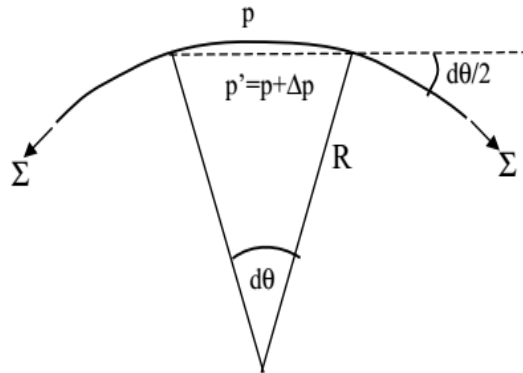
1.12 Surface Tension

- Notation: Σ [Tension force / Length] \equiv [Surface energy / Area].
- Surface tension is due to the intermolecular attraction forces in the fluid.
- At the interface of two fluids, surface tension implies in a pressure jump across the interface. Σ gives rise to Δp across an interface.
- For a water/air interface: $\Sigma = 0.07 \text{ N/m}$. This is a function of temperature, impurities etc. . .
- 2D Example:

$$\underbrace{\cos \frac{d\theta}{2}}_{\approx 1} \cdot \Delta p \cdot R d\theta = 2 \Sigma \underbrace{\sin \frac{d\theta}{2}}_{\approx \frac{d\theta}{2}} \approx 2 \Sigma \frac{d\theta}{2}$$

$$\therefore \Delta p = \frac{\Sigma}{R}$$

Higher curvature implies in higher pressure jump at the interface.



- 3D Example: Compound curvature

$$\Delta p = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \Sigma$$

where R_1 and R_2 are the principle radii of curvature.

1.13 Body Forces – Gravity

- Conservative forces

$$\vec{F} = -\nabla\varphi \text{ for some } \varphi,$$

where φ is the force potential.

$$\oint \vec{F} \cdot d\vec{x} = 0 \text{ or } \int_1^2 \vec{F} \cdot d\vec{x} = - \int_1^2 \nabla\varphi \cdot d\vec{x} = \varphi(\vec{x}_1) - \varphi(\vec{x}_2)$$

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- A special case of a conservative force is gravity $\vec{F} = -\rho g \hat{k}$.
 - In this case the gravitational potential is given by $\varphi_g = \rho g z$. Therefore:

$$\boxed{\vec{F} = \nabla(-\varphi_g) = \underbrace{\nabla(-\rho g z)}_{\equiv \text{hydrostatic pressure } p_s} = \nabla p_s}$$

- Substitute in Navier-Stokes equation

$$\begin{aligned} \rho \frac{D\vec{v}}{Dt} &= -\nabla p + \underbrace{\vec{F}}_{\text{body force}} + \rho \nu \nabla^2 \vec{v} \\ &= -\nabla p + \nabla(-\rho g z) + \rho \nu \nabla^2 \vec{v} \end{aligned}$$

- Define: total pressure \equiv hydrostatic pressure + hydrodynamic pressure

$$\begin{aligned} p &= p_s + p_d \\ p &= -\rho g z + p_d \quad \implies \end{aligned}$$

$$\boxed{p_d = p + \rho g z}$$

- Re-write Navier-Stokes:

$$\begin{aligned} \rho \frac{D\vec{v}}{Dt} &= -\nabla \left(\underbrace{p + \rho g z}_{p_d = p + \rho g z} \right) + \rho \nu \nabla^2 \vec{v} \\ \rho \frac{D\vec{v}}{Dt} &= -\nabla p_d + \rho \nu \nabla^2 \vec{v} \end{aligned}$$

Therefore:

- Presence of gravity body force is equivalent to replacing the total pressure by a dynamic pressure ($p_d = p - p_s = p + \rho g z$) in the Navier-Stokes(N-S) equation.
- Solve the N-S equation with p_d . To calculate the total pressure p simply add back the hydrostatic component $p = p_d + p_s = p_d - \rho g z$.