More on intersecting families

A family $\mathcal{A} \subseteq \mathcal{P}(n)$ is *t-intersecting* if $|A \cap B| \geq t$, for all $A, B \in \mathcal{A}$.

When n is large enough, the Erdős-Ko-Rado Theorem generalises as follows.

Theorem 23. *Let $1 < t \leq k$ be positive integers. There exists an integer $n_0 = n_0(k,t)$ such that the following holds. For all $n > n_0$, if $\mathcal{A} \subseteq [n]^{(k)}$ is t-intersecting, then*

$$|\mathcal{A}| \leq \binom{n-t}{k-t},$$

with equality if and only if $\mathcal{A}$ is of the form $\{A \in [n]^{(k)} : T \subseteq A\}$, where T is a t-element subset of $[n]$.

Proof. We may assume that $\mathcal{A}$ is maximal and so there are $A, B \in \mathcal{A}$ with $|A \cap B| = t$ (exercise).

Let us fix $A, B \in \mathcal{A}$ with $|A \cap B| = t$. If $A \cap B \subseteq C$, for all $C \in \mathcal{A}$, then $|\mathcal{A}| \leq \binom{n-t}{k-t}$ as required.

So suppose that there exists $C \in \mathcal{A}$ with $A \cap B \nsubseteq C$. Every $D \in \mathcal{A}$ must have at least $t+1$ elements in $A \cup B \cup C$. Thus

$$|\mathcal{A}| \leq \binom{|A \cup B \cup C|}{t+1}\binom{n}{k-t-1} \leq (3k)^{t+1} n^{k-t-1} < \binom{n-t}{k-t},$$

provided n is large enough. ■

What if we allow only intersection of a fixed size?

Theorem 24 (Fisher's Inequality)**.** *Let $k \geq 1$. Suppose that $\mathcal{A} \subseteq \mathcal{P}(n)$ satisfies $|A \cap B| = k$ for all distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq n$.*

Proof. If there exists $A \in \mathcal{A}$ with $|A| = k$, then $A \subseteq B$ for all $B \in \mathcal{A}$, and the family $\{B \setminus A : B \in \mathcal{A}\}$ consists of pairwise disjoint sets, and so $|\mathcal{A}| \leq n$.

Otherwise, we may assume that $|A| > k$ for all $A \in \mathcal{A}$. Let $\chi_A \in \mathbb{R}^n$ denote the characteristic vector of A, so

$$\chi_A(i) = \begin{cases} 1 & i \in A \\ 0 & i \notin A. \end{cases}$$

We claim that the vectors in $\{\chi_A : A \in \mathcal{A}\}$ are linearly independent. Then $|\mathcal{A}| = |\{\chi_A : A \in \mathcal{A}\}| \leq \dim(\mathbb{R}^n) = n$.

So suppose that $\sum\limits_{A \in \mathcal{A}} \lambda_A \chi_A = 0$. For any $B \in \mathcal{A}$, since $\langle \chi_A, \chi_B \rangle = |A \cap B| = k$ for $A \neq B$ and $\langle \chi_B, \chi_B \rangle = |B|$, we have

$$0 = \langle \sum_{A \in \mathcal{A}} \lambda_A \chi_A, \chi_B \rangle = \lambda_B |B| + \sum_{A \neq B} \lambda_A k = \lambda_B(|B| - k) + \Lambda k,$$

where $\Lambda = \sum\limits_{A \in \mathcal{A}} \lambda_A$. Thus (noting that $|B| > k$)

$$\lambda_B = -\frac{k\Lambda}{|B| - k}.$$

If $\Lambda = 0$, then $\lambda_B = 0$ for all $B \in \mathcal{A}$. If $\Lambda \neq 0$, then Λ and λ_B have opposite sign. But this holds for any $B \in \mathcal{A}$, which is impossible since $\Lambda = \sum\limits_{B \in \mathcal{A}} \lambda_B$. We conclude that the vectors χ_A are linearly independent, as required. ■

We continue with *modular* intersection theorems, where the allowed sizes of intersections are specified modulo p. Here is our first example.

Theorem 25. (Oddtown Theorem) *Let $\mathcal{A} \subseteq \mathcal{P}(n)$ be a family such that*

- *$|A|$ is odd for all $A \in \mathcal{A}$*
- *$|A \cap B|$ is even for all distinct $A, B \in \mathcal{A}$.*

Then $|\mathcal{A}| \leq n$.

First proof of the Oddtown Theorem. We work over the field with two elements $\mathbb{F}_2$. We identify each element of with its characteristic vector in $\mathbb{F}_2^n$.

Then, for all $A, B \in \mathcal{A}$, we have

$$\langle \chi_A, \chi_B \rangle = |A \cap B| = \begin{cases} 0 & if A \neq B \\ 1 & A = B. \end{cases}$$

We claim that $\{\chi_A : A \in \mathcal{A}\}$ is linearly independent in $\mathbb{F}_2^n$. If $\sum_{A\in\mathcal{A}} \lambda_A\chi_A = 0$, then, for all $B \in \mathcal{A}$, we have

$$0 = \langle \sum_{A\in\mathcal{A}} \lambda_A\chi_A, \chi_B\rangle = \lambda_B.$$

Hence the vectors $\{\chi_A : A \in \mathcal{A}\}$ are linearly independent and so $|\mathcal{A}| \leq n$. ■

Proof 2 of the Oddtwon Theorem. We work over the field with two elements $\mathbb{F}_2$. Let $\mathcal{A} = \{A_1, \dots, A_m\}$.

Let $M = (m_{ij})$ be the $m \times n$ incidence matrix where

$$m_{ij} = \begin{cases} 1 & j \in A_i \\ 0 & j \notin A_i. \end{cases}$$

Then $N = MM^T$ is the $m \times m$ identity matrix. But then $\text{rank}(N) = m = |\mathcal{A}| \leq \text{rank}(M) \leq n$. ■

For the next theorem, we need to introduce the *multilinearization trick*. Given a polynomial f in one or more variables, we define $\tilde{f}$ to be the polynomial obtained by replacing every occurrence x^i, $i > 1$, of each variable x by just x^1. For instance if $f(x, y, z) = 4x^3+xy+z^{10}$ then $\tilde{f}(x, y, z) = 4x+xy+z$. Here is the crucial observation: if we evaluate f and $\tilde{f}$ at a point where all variables take values 0 or 1 then they both take the same value.

Here is good example of this trick in action.

Theorem 26 (Modular Frankl-Wilson Theorem). *Let p be prime and $S \subseteq \{0, 1, \dots, p-1\}$. Suppose that $\mathcal{A} \in \mathcal{P}(n)$ satisfies:*

- $|A| \notin S \bmod p$, *for all* $A \in \mathcal{A}$;
- $|A \cap B| \in S \pmod{p}$, *for all distinct* $A, B \in \mathcal{A}$.

Then

$$|\mathcal{A}| \leq \sum_{i=0}^{|S|} \binom{n}{i}.$$

Proof. We work over the field with prime number of elements $\mathbb{F}_p$, and introduce variables $x = (x_1, \dots, x_n)$. For each $A \in \mathcal{A}$, we define the polynomial

$$f_A(x) = \prod_{s\in S}\left(\sum_{i\in A} x_a - s\right).$$

Then, for $B \in \mathcal{A}$, if $B \neq A$ we have

$$f_A(\chi_B) = \prod_{i\in A}(|A\cap B| - s) = 0,$$

while if $A = B$ we have

$$f_A(\chi_B) = \prod_{i\in A}(|A| - s) \neq 0.$$

We now replace each polynomial f_A by the corresponding multilinear polynomial $\tilde{f}_A$. For any $\chi \in \{0,1\}^n$, we have

$$\tilde{f}_A(\chi) = f_A(\chi).$$

It follows that, for $B \in \mathcal{A}$, if $B \neq A$ we have

$$\tilde{f}_A(\chi_B) = \prod_{i\in A}(|A\cap B| - s) = 0,$$

while if $A = B$ we have

$$\tilde{f}_A(\chi_B) = \prod_{i\in A}(|A| - s) \neq 0.$$

The polynomials $\{\tilde{f}_A(x) : A \in \mathcal{A}\}$ are linearly independent. For if $\sum \lambda_A \tilde{f}_A = 0$ then, for any $B \in \mathcal{A}$,

$$0 = \left(\sum_{A\in\mathcal{A}} \lambda_A \tilde{f}_A\right)(\chi_B) = \lambda_B.$$

The $\tilde{f}_A$ are therefore linearly independent, and lie in the space of multilinear polynomials of dgeree at most $|S|$. This has dimension

$$\sum_{i=0}^{|S|}\binom{n}{i}$$

(it is spanned by the monomials $\{\prod_{i\in A} x_i : |A| \leq i\}$). Thus $|\mathcal{A}| \leq \sum_{i=0}^{|S|}\binom{n}{i}$. ■

What if we drop the modular constraint?

For a set $S \subseteq \mathbb{N}$, we say that a family $\mathcal{A}$ is *S-intersecting* if $|A\cap B| \in S$ for all distinct $A, B \in \mathcal{A}$.