Fundamental Equations in terms of the substantial derivative:

Let's consider the continuity eq:

$$\frac{\partial \rho}{\partial t} + \Delta.(\rho \vec{V}) = 0$$

Velocity identity: $\Delta.(\rho \vec{V}) = \rho \nabla.\vec{V} + \vec{V}.\nabla \rho$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \Delta.(\rho \vec{V}) + \rho \nabla.\vec{V} + \vec{V}.\nabla \rho$$

The continuity eq becomes

$$\frac{\partial \rho}{\partial t} + \rho \nabla.\vec{V} = 0$$

Momentum eq:

$$\frac{\partial}{\partial t}(\rho u) + \Delta.(\rho u \vec{V}) = -u\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{vis}$$

$$(1) \;\; \frac{\partial}{\partial t}(\rho u) = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

$$(2) \;\; \Delta.(\rho u \vec{V}) = u \Delta.(\rho \vec{V}) + \rho \vec{V}.\Delta.u$$

$$(1) + (2) = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + u \Delta.(\rho \vec{V}) + \rho \vec{V}.\Delta.u$$

$$= u\left[\frac{\partial \rho}{\partial t} + \Delta.(\rho \vec{V})\right] + \rho \frac{\partial u}{\partial t} + \rho \vec{V}.\Delta.u$$

$$= 0 + \rho \frac{\partial u}{\partial t} + \rho \vec{V}.\Delta.u$$

$$= \rho \frac{\partial u}{\partial t} + \rho \vec{V}.\Delta.u$$

$$= \rho\left[\frac{\partial u}{\partial t} + (V.\vec{\nabla} u)\right]$$

$$= \rho\left[\frac{Du}{Dt}\right]$$

Then the x-component of the momentum eq becomes:

$$\rho\left[\frac{Du}{Dt}\right] = -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{vis}$$

Similarly, the y- and z- component

$$\rho\left[\frac{Dv}{Dt}\right]=-\frac{\partial p}{\partial y}+\rho f_y+\left(F_y\right)_{vis}$$

$$\rho\left[\frac{Dw}{Dt}\right]=-\frac{\partial p}{\partial z}+\rho f_z+\left(F_z\right)_{vis}$$

<u>The energy Equation</u>:

$$\frac{\rho D\left(\rho+\frac{v^2}{2}\right)}{Dt}=\rho\dot{q}+\text{-}\nabla.(p\vec{V})+\rho(\vec{f}.\vec{V})+\dot{W}_{viscous}+\dot{Q}_{viscous}$$

Pathlines, Streamlines and Streaklines of Flow:

pathline: it is the path that the fluid element moves along.

Streamline is defined as the line whose tangent at any point is in the direction of the velocity vector at that point.
If the flow is unsteady, the streamline pattern is different at different instants of time.

If the flow is steady, then streamlines and pathlines are identical.

What is Streakline?

Streakline is defined as the locus of fluid elements which have passed through a point.
For steady flow, pathlines, streamlines and streaklines are all the same curves.

Equation for a streamline:

Let's pick up a certain point in the flow and let $d\vec{S}$ be the vector pointing in the same directions the velocity vector at that point:

$$d\vec{S}=dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k}$$

$$\vec{V}=u\,\vec{i}+v\,\vec{j}+w\,\vec{k}$$

By definition, the streamline is the curve whose tangent is in the direction of the velocity vector, this means that $\vec{V}$ and $d\vec{S}$ are in the same direction, the cross product of tow vectors parallel to each other is:

$$d\vec{S}x\vec{V}=\left|d\vec{S}\right|\left|\vec{V}\right|\sin\theta=0$$

$$\Rightarrow d\vec{S}x\vec{V}=0$$

$$d\vec{S}x\vec{V} = \begin{bmatrix} i & j & k \\ dx & dy & dz \\ u & v & \omega \end{bmatrix} = (\omega dy - vdz)\vec{i} - (\omega dx - udz)\vec{j} + (vdx - udy)\vec{k} = 0$$

This means that:

$$\omega dy - vdz = 0$$

$$\omega dx - udz = 0$$

$$vdx - udy = 0$$

If the floe is tow dimensional, ω=0 , z=1

$$vdz = udy \Rightarrow \frac{dy}{dx} = \frac{v}{u}$$

This means that the slope of the streamline at a point is the ratio of the velocity, given by $\frac{v}{u}$

Example: Given a flow field with $u = \frac{y}{(x^2 + y^2)}$ and $v = \frac{-x}{(x^2 + y^2)}$

Find the equation of the streamline passing through the point (0, 5)?

Solution:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{\frac{-x}{(x^2 + y^2)}}{\frac{y}{(x^2 + y^2)}} = \frac{-x}{y}$$

$$ydy = -xdx$$

$$\int ydy = -\int xdx$$

$$\Rightarrow \frac{y^2}{2} + C_1 = \frac{-x^2}{2} + C_2 \Rightarrow x^2 + y^2 = 2(C_2 - C_1) = C$$

$$\Rightarrow x^2 + y^2 = C$$

to find C: we use the point (0,5)

$$\Rightarrow 0^2 + 5^2 = C \Rightarrow C = 25$$

$\Rightarrow$ the equation of the streamline is given by:

$$x^2 + y^2 = 25$$

Angular Velocity, Vorticity and Strain;

Let's consider a fluid element moving through the flow field, this element translates, may rotate and its shape may be distorted.

Let's consider a fluid element in two dimensional flow:

Consider point A:

Distance in x-direction that A moves during time increment $\Delta t = v\Delta t$

Distance in y-direction that A moves during time increment $\Delta t = \left(v + \frac{dv}{dx}dx \right)\Delta t$

Net displacement in y-direction of C relative to A

$$= \left(v + \frac{dv}{dx}dx \right)\Delta t - v\Delta t \quad = \left(\frac{dv}{dx}dx \right)\Delta t$$

$$\tan(\Delta\theta_2) = \frac{\left[\left(\frac{dv}{dx}dx \right) \right]\Delta t}{dx} = \left(\frac{dv}{dx} \right)\Delta t$$

if $\Delta\theta_2$ is small $\Rightarrow \tan(\Delta\theta_2) \approx \Delta\theta_2 \quad \Rightarrow \Delta\theta_2 = \left(\frac{dv}{dx} \right)\Delta t$

Similarly for point B:

$$\tan(-\Delta\theta_1) = \left(\frac{du}{dy} \right)\Delta t$$

If $\Delta\theta_1$ is small $\Rightarrow \tan(-\Delta\theta_1) \approx -\Delta\theta_1 \quad \Rightarrow \Delta\theta_1 = -\left(\frac{du}{dy} \right)\Delta t$

Then *__the angular velocities__* of lines AB and AC are:

AB: $$\frac{d\theta_1}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = -\frac{du}{dy}$$

AC: $$\frac{d\theta_2}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_2}{\Delta t} = -\frac{dv}{dx}$$

Then the angular velocity of the fluid element is the average of both angular velocities

$$\omega_z = \frac{1}{2}\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt} \right) = \frac{1}{2}\left(\frac{-du}{dy} + \frac{dv}{dx} \right) = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy} \right)$$

Similarly for yz and zx planes:

$$\omega_x = \frac{1}{2}\left(\frac{d\omega}{dy} - \frac{dv}{dz} \right)$$

$$\omega_y = \frac{1}{2}\left(\frac{du}{dz} - \frac{d\omega}{dx} \right)$$

$$\Rightarrow \vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

Vorticity:

$$\vec{\zeta} = 2\vec{\omega}$$

$$\Rightarrow \vec{\zeta} = \left(\frac{d\omega}{dy} - \frac{dv}{dz}\right)\vec{i} + \left(\frac{du}{dz} - \frac{d\omega}{dx}\right)\vec{j} + \left(\frac{dv}{dx} - \frac{du}{dy}\right)\vec{k}$$

Mathematically, $\zeta = \Delta x \vec{V}$ the curl of velocity is equal to the vorticity.

If $\Delta x \vec{V} \neq 0$ at every point in a flow, the flow is called rotational.
If $\Delta x \vec{V} = 0$ at every point in a flow, the flow is called irrotational.
All viscous flows are rotational.

Let's consider a two-dimensional flow:

$$\Rightarrow \vec{\zeta} = \left(\frac{dv}{dx} - \frac{du}{dy}\right)\vec{k}$$

if the flow is irrotational, $\frac{dv}{dx} = \frac{du}{dy}$

Strain:
Let's retune to the fluid element. The strain is defined as the change in the angle κ

or change in the angle $\kappa = \kappa_{\Delta t} - \kappa_t = 90 - \Delta\theta_2 + \Delta\theta_1 + 90$

$$\Rightarrow \Delta\kappa = \Delta\theta_1 - \Delta\theta_2$$

Strain is (+ve) if κ is decreasing.
Strain= $-\Delta\kappa = \Delta\theta_2 - \Delta\theta_1$

Let's denote the time rate of strain by ε_{xy}

$$\varepsilon_{xy} = \frac{strain}{time} = \frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}$$

what is $\frac{d\theta_2}{dt}$? $\frac{d\theta_2}{dt} = \frac{dv}{dx}$

and $\frac{d\theta_1}{dt} = -\frac{du}{dy}$

$$\varepsilon_{xy} = \frac{dv}{dx} + \frac{du}{dy}$$

Similarly,

$$\varepsilon_{yz} = \frac{d\omega}{dy} + \frac{dv}{dz}$$

$$\varepsilon_{zx} = \frac{du}{dz} + \frac{d\omega}{dx}$$

Circulartion:

Later in the course, we will find that the lift of a wing is a function of the circulation (Joukowski Theorem).
Mathematically, the circulation Γ is

$$\Gamma = -\oint_C \vec{V}.dS$$

Γ (+ve) if it is clockwise.
Γ (-ve) if it is counterclockwise.

The circulation can be related to vorticity. Using Stoke's Theorem:

$$\Gamma = -\oint_C \vec{V}.dS = -\iint_S (\nabla x\vec{V}).d\vec{S}$$

If the flow is irrotational, $\Rightarrow \nabla x\vec{V} = 0 \Rightarrow \Gamma = 0$
we can realte the verticity to the circulation. If C is infinitesimally small, then:

$$d\Gamma = -(\nabla x\vec{V}).d\vec{S} = -(\nabla x\vec{V})\vec{n}.dS$$

$$\Rightarrow (\nabla x\vec{V})\vec{n} = -\frac{d\Gamma}{dS}$$

Stream Function:
Recall, the differential equation for a two dimensional flow is:

$$\frac{dy}{dx} = \frac{v}{u}$$

Integrating this equation yields:

f(x,y)=C

where

C is a constant of integration.
C has different values from one streamline to another. We can write the above equation as: $\vec{\Psi}(x, y) = C$

Where Ψ is called *the stream Function.*

Let's consider a two streamline:

$$\Delta\vec{\Psi} = C_2 - C_1$$

Let's assume that these streamline are closer to each other and the distance between them is very small Δn such that the velocity is constant across Δn.

This becomes like streamtube. The mass flow rate is:

$$\Delta\vec{\Psi} = \rho VA = \rho V(\Delta n).1$$

$$\frac{\Delta\vec{\Psi}}{\Delta n} = \rho V$$

Taking lim of this equation, we have

$$\lim_{\Delta n \to 0} \frac{\Delta\vec{\Psi}}{\Delta n} = \frac{d\vec{\Psi}}{dn} = \rho V$$

The mass flow rate is the sum of the mass through Δy and $-\Delta x$

$$\rho V = massflow = \rho u\Delta y + \rho v(-\Delta x)$$

In the limit, this becomes:

$$d\vec{\Psi} = \rho udy - \rho vdx$$

The chain rule fives:

$$d\vec{\Psi} = \frac{d\vec{\Psi}}{dx}dx - \frac{d\vec{\Psi}}{dy}dy$$

Comparing both equation yields:

$$\rho u = \frac{d\vec{\Psi}}{dy}$$

$$\rho v = -\frac{d\vec{\Psi}}{dx}$$

For incompressible flow, ρ is constant $\Rightarrow \Psi = \frac{\vec{\Psi}}{\rho} = \text{constant}$

$$\Rightarrow u = \frac{d\left(\frac{\vec{\Psi}}{\rho}\right)}{dy} = \frac{d\vec{\Psi}}{dy}$$

$$\Rightarrow v = -\frac{d\vec{\Psi}}{dx}$$

Velocity Potential:

Recall that for irrotational flow, we have $\vec{\zeta} = \nabla x \vec{V} = 0$

By vector identity: $\nabla x(\nabla \phi) = 0$

So for the flow to be irrotational, the velocity filed is given by $\vec{V} = \nabla \phi$

$$u\,\vec{i} + v\,\vec{j} + \omega\,\vec{k} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$\Rightarrow u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad \omega = \frac{\partial \phi}{\partial z}$$