review of Vector Relation:

Consider two vectors:

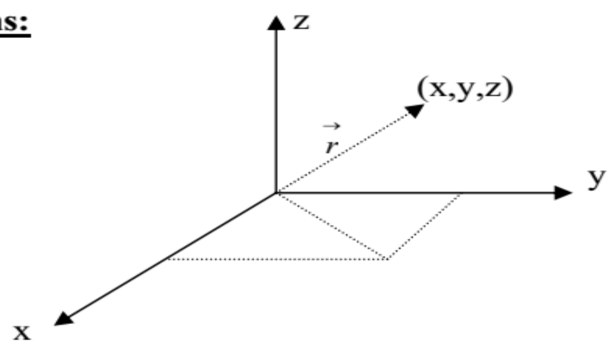
Vector addition: $\vec{A}+\vec{C}=\vec{B}$

Vector subtraction: $\vec{B}-\vec{C}=\vec{A}$

Scale product (Dot product): $A \bullet B = |A||B|\cos\theta$ =scalar

Vector product (cross product): $AxB = (|A||B|\sin\theta)\vec{e}$ = vector product

Coordinate Systems:



$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

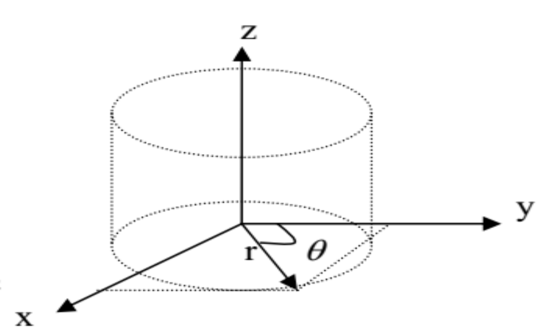
or you can write it as

$$\vec{r} = r_x\,\vec{i} + r_y\,\vec{j} + r_z\,\vec{k}$$

Where

$$r_x = x$$
$$r_y = y$$
$$r_z = z$$

Cylindrical Coordinate:



$x = r\cos\theta$

$y = \sin\theta$

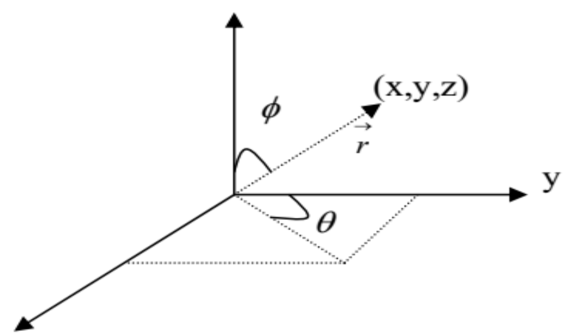
$z = z$

$$\Rightarrow \quad x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$
$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$
z=z

Spherical Coordinate:



$$x = (r\sin\phi)\cos\theta = r\cos\theta\sin\phi$$
$$y = (r\sin\phi)\sin\theta = r\sin\theta\sin\phi$$
$$z = r\cos\phi = r\cos\theta$$

Recall,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \Rightarrow r^2 = x^2 + y^2 + z^2$$

Proof:

$$\vec{r} = r\cos\theta\sin\phi\,\vec{i} + r\sin\theta\sin\phi\,\vec{j} + r\cos\theta\,\vec{k}$$
$$\vec{r}.\vec{r} = r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$
$$\vec{r}.\vec{r} = r^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) + r^2\cos^2\theta$$
$$\vec{r}.\vec{r} = r^2 = r^2\sin^2\phi + r^2\cos^2\theta$$
$$\Rightarrow r^2 = r^2$$

scalar and Vector Fields:
scalar field can be expressed in terms of the coordinate system used.
Example: pressure $p(x, y, z, t)$
Temperature $t(x, y, z, t)$ for constant coordinate
Time t
Density $\rho(x, y, z, t)$

Vector field can also be expressed in terms of coordinate system
Example: velocity $v = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$

$$\text{Where } \begin{aligned} v_x &= v_x(x,y,z,t) \\ v_y &= v_y(x,y,z,t) \\ v_z &= v_z(x,y,z,t) \end{aligned}$$

Scalar and Vector Products:

Suppose we have tow vectors

$$\vec{A} = A_x\,\vec{i} + A_y\,\vec{j} + A_z\,\vec{k}$$

$$\vec{B} = B_x\,\vec{i} + B_y\,\vec{j} + B_z\,\vec{k}$$

Scalar product (dot product):

$$A \bullet B = A_x B_x + A_y B_y + A_z B_z\,;$$

$$\vec{i}\,.\,\vec{i} = \vec{j}\,.\,\vec{j} = \vec{k}\,.\,\vec{k} = 1$$

$$\vec{i}\,.\,\vec{j} = \vec{i}\,.\,\vec{k} = \vec{k}\,.\,\vec{i} = 0$$

$$\vec{j}\,.\,\vec{i} = \vec{j}\,.\,\vec{k} = \vec{k}\,.\,\vec{j} = 0$$

Vector product(cross product)

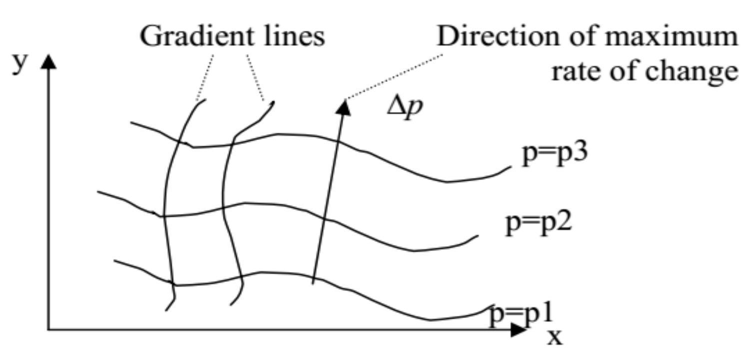
$$AxB = \begin{bmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \left(A_y B_z - A_z B_y\right)\vec{i} - \left(A_x B_z - A_z B_x\right)\vec{j} + \left(A_x B_y - A_y B_x\right)\vec{k}$$

Similar thing can be done for the other coordinate systems.

Gradient and Scalar Filed:

Suppose we have a scalar field i.e. pressure $p(x, y, z, t)$

The gradient of p is a vector with its magnitude being the maximum rate of change per unit length of the coordinate system at a given point and its direction is the maximum rate of change of p at a given point.



Mathematically, the gradient of p is

$$\Delta p = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}$$

<u>Divergence and a Vector Field:</u>

Given a vector field i.e. velocity field

$$\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$$

the divergence of $\vec{v}$ is

$$\Delta \bullet \vec{v} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}$$

we will proof shortly that the divergence is the time rate of change of the volume of a moving element per unit volume.

$$\int \vec{v} \bullet n ds = \int \left|\vec{v}\right| \left|n\right| \cos\theta ds = \left|\vec{v}\right| \left|n\right| \cos\theta \int ds = \left|\vec{v}\right| \cos\theta ds$$

<u>Curl of Vector field:</u>

The curl of a vector field $\vec{v}$ is

$$\text{Curl of } \vec{v} = \Delta x\, \vec{v} = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{bmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{i} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\vec{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{k}$$

We will proof that in this chapter that the angular velocity ω of a fluid element is $\frac{1}{2} curl(\vec{v})$