Strategic Form Games

Games

Games can be categorized in to following two forms as below. We will start here with the first category and postpone the discussion of the second category for later.

1. Strategic Form Games (also called Normal Form Games)

2. Extensive Form Games

Strategic Form Games

1. Let $\mathcal{I} = \{1, \ldots, I\}$ be a finite set of players, where $I \in \mathbb{N}$ is the number of players.

2. Let $S_i (i \in \mathcal{I})$ be the (finite) set of *pure strategies* available to player $i \in \mathcal{I}$.

3. $$S = S_1 \times S_2 \times \cdots \times S_I$$

   (Cartesian product of the pure strategies) = Set of pure strategy profiles.

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Conventions

We write, $s_i \in S_i$ for a pure strategy of player i. We also write, $s = (s_1, s_2, \ldots, s_I) \in S$ for a pure strategy profile.

"$-i$" denotes the player i's "opponents" and refers to all players other than some given player i. Thus, we can write, $S_{-i} = \times_{j \in \mathcal{I}, j \neq i} S_j$

Just as before, $s_{-i} \in S_{-i}$ denotes a pure strategy profile for the opponents of i. Hence,

$$s = (s_i, s_{-i}) \in S,$$

is a pure strategy profile.

$u_i : S \to \mathbb{R} =$ Pay-off function (real-valued function on S) for player i.

$u_i(s) =$ *von Neumann-Morgenstern utility* of player i for each profile $s = (s_1, s_2, \ldots, s_I)$ of pure strategies.

Definition *A strategic form game is a tuple*

$$(\mathcal{I}, \{S_1, S_2, \ldots, S_I\}, \{u_1, u_2, \ldots, u_I\})$$

consisting of a a set of players, pure strategy spaces *and* pay-off functions.

Definition *A two-player zero-sum game is a strategic form game with* $\mathcal{I} = \{1, 2\}$ *such that*

$$\forall_{s \in S} \quad \sum_{i=1}^{2} u_i(s) = 0.$$

Definition *A mixed strategy set for player i, Σ_i is the set of probability distributions over the pure strategy set S_i*

$$\Sigma_i = \left\{ \sigma_i : S_i \to [0,1] \mid \sum_i \sigma_i(s_i) = 1 \right\}.$$

The space of mixed strategy profile $= \Sigma = \times_{i \in \mathcal{I}} \Sigma_i$.

As before, we write: $\sigma_i \in \Sigma_i$, and $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_I\} \in \Sigma$.

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The *support* of a mixed strategy σ_i = The set of pure strategies to which σ_i assigns positive probability.

Player i's pay-off to profile σ is

$$\begin{aligned} u_i(\sigma) &= \mathbb{E}_{\sigma_i} u_i(\cdot, \sigma_{-i}) \\ u_i(\sigma) &= u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \\ u_i(s_i, \sigma_{-i}) &= \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \Bigg(\prod_{j \neq i} \sigma_j(s_j) \Bigg) u_i(s_i, s_{-i}). \end{aligned}$$

Hence,

$$\begin{aligned} u_i(\sigma) &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i) \Bigg(\prod_{j \neq i} \sigma_j(s_j) \Bigg) u_i(s_i, s_{-i}) \\ &= \sum_{s \in S} \Bigg(\prod_j \sigma_j(s_j) \Bigg) u_i(s). \end{aligned}$$

Domination & Nash Equilibrium

Definition *A pure strategy s_i is* strictly dominated *for player i if*

$$\exists_{\sigma_i' \in \Sigma_i} \forall_{s_{-i} \in S_{-i}} u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}).$$

A pure strategy s_i is weakly dominated *for player i if*

$$\exists_{\sigma_i' \in \Sigma_i} \Bigg(\forall_{s_{-i} \in S_{-i}} u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i}) \\ \wedge \quad \exists_{s_{-i} \in S_{-i}} u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \Bigg).$$

Definition **Best Response:** *The set of best responses for player i to a pure strategy profile $s \in S$ is*

$$BR_i(s) = \Bigg\{ s_i^* \in S_i | \forall_{s_i \in S_i} \; u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \Bigg\}.$$

Let the joint best response set be $BR(s) = \times_i BR_i(s)$.

Definition **Nash Equilibrium:** *A pure strategy profile s^* is a* Nash equilibrium *if for all players i,*

$$\forall_{s_i \in S_i}\ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Thus a Nash equilibrium is a strategy profile s^* such that $s^* \in BR(s^*)$.

A Nash equilibrium s^* is strict *if each player has a unique best response to his rivals' strategies: $BR(s^*) = \{s^*\}$.*

$$\forall_{s_i \neq s_i^*} u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*).$$

A mixed strategy profile σ^* is a Nash equilibrium *if for all players i,*

$$\forall_{s_i \in S_i}\ u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*).$$

Remark: Since expected utilities are "linear in the probabilities," if a player uses a non-degenerate mixed strategy in a Nash equilibrium (non-singleton support), he must be indifferent between all pure strategies to which he assigns positive probability. (It suffices to check that no player has a profitable pure-strategy deviation).

Example

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

For column-player, M is dominated by R. Column-player can eliminate M from his strategy space. The pay-off matrix reduces to

New Pay-offs

	L	R
U	4,3	6,2
M	2,1	3,6
D	3,0	2,8

For row-player, M and D are dominated by U. Row-player can eliminate M and D. The new pay-off matrix is

New Pay-offs

	L	R
U	4,3	6,2

Next, column-player eliminates R as it is dominated by U and reduces the pay-off matrix to

New Pay-offs

	L
U	4,3

Note that

$$BR_r(U, L) = U, \quad \& \quad BR_c(U, L) = L, \quad \& \quad BR(U, L) = (U, L).$$

(U, L) is a strict Nash equilibrium.
Remark: *Mixed Strategy* (Not a Nash equilibrium.)