Nash Equilibrium

Fixed Point Theorems

Definition *A point $x \in K$ is a* fixed point *of an injective function $f: K \to K$, if*

$$x = f(x).$$

Definition *A point $x \in K$ is a* fixed point *of a mapping $\Psi: K \to 2^K$, if*

$$x \in \Psi(x).$$

Theorem **Brouwer's Fixed Point Theorem***: If $f : K \to K$ is a* continuous function *from a nonempty, compact, convex subset K of a finite dimensional TVS (topological vector space) into itself, then f has a fixed point, i.e.,*

$$\exists_{x \in K}\ x = f(x).$$

Theorem **Kakutani's Fixed Point Theorem***: If $\Psi : K \to 2^K$ is a convex-valued, uhc (upper hemi-continuous) map from a nonempty, compact, convex subset K of a finite dimensional TVS to the nonempty subsets of K, then Ψ has a fixed point, i.e.,*

$$\exists_{x \in K}\ x \in \Psi(x).$$

Definition *Topological Vector Space: L = vector space with a T_1 topology*

$$(\forall_{x \neq y \in L}\ \exists_{G_x = \text{ open set}}\ x \in G_x \wedge y \notin G_x)$$

which admits continuous vector space operations.

Example: $\mathbb{R}^n$ with standard Euclidean topology. (Only instance of a finite dimensional TVS.)

Theorem Existence of a Mixed Strategy Equilibrium *(Nash 1950). Every finite strategic-form game has a mixed strategy equilibrium.*

Proof: Player i's *reaction correspondence*, Ψ_i, maps each strategy profile σ to the set of mixed strategies that maximize player i's pay-offs when his rivals play σ_{-i}:

$$\Psi_i(\sigma) = \left\{\sigma_i' \mid \forall_{s_i \in S_i}\ u_i(\sigma_i', \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})\right\}.$$

Thus,

$$\Psi_i : \Sigma \to 2^{\Sigma_i}.$$

Define

$$\Psi : \Sigma \to 2^{\Sigma} : \sigma \mapsto \times_i \Psi_i(\sigma).$$

Thus this *correspondence map* is the Cartesian product of Ψ_i's.

A fixed point of Ψ (if exists) is a σ^* such that

$$\sigma^* \in \Psi(\sigma^*).$$

Note that

$$\forall_{s_i \in S_i}\ u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*),$$

by definition. Thus a fixed point of Ψ provides a mixed strategy equilibrium σ^*.

Claims:

1. Σ = Nonempty, compact and convex subset of a TVS.

   $\Sigma_i = \Delta_{|S_i|-1} = |S_i| - 1$ dimensional simplex, since

   $$\Sigma_i = \left\{(\sigma_{i,1}, \ldots, \sigma_{i,|S_i|}) \mid \sigma_{i,j} \geq 0, \sum_j \sigma_{i,j} = 1\right\}.$$

   Rest follows since $\Sigma = \times_i \Sigma_i$.

2. $u_i =$ Linear Function.

$$\forall_{0<\lambda<1}\ u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) = \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda) u_i(\sigma_i'', \sigma_{-i}).$$

Hence u_i is a continuous function in his own mixed strategy. Since Σ is compact, u_i attains maxima in Σ.

$$\forall_{\sigma\in\Sigma}\ \Psi(\sigma) \neq \emptyset.$$

3.

$$\forall_{\sigma\in\Sigma}\ \Psi(\sigma) = \text{ convex.}$$

Let σ_i', $\sigma_i'' \in \Psi(\sigma)$. By definition,

$$\forall_{s_i\in S_i}\ (u_i(\sigma_i', \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})) \wedge (u_i(\sigma_i'', \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})).$$

Hence

$$\forall_{0<\lambda<1}\ \forall_{s_i\in S_i}\ u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}),$$

and

$$\forall_{0<\lambda<1}\ \lambda\sigma_i' + (1-\lambda)\sigma_i'' \in \Psi_i(\sigma).$$

4. $\Psi =$ uhc. Consider a sequence

$$\left\{(\sigma^n, \hat{\sigma}^n) \mid \hat{\sigma}^n \in \Psi(\sigma^n)\right\}_n.$$

We wish to show that

$$\text{If} \quad \lim_{n\to\infty} (\sigma^n, \hat{\sigma}^n) = (\sigma, \hat{\sigma}) \quad \text{then} \quad \hat{\sigma} \in \Psi(\sigma).$$

Suppose Not! Then

$$\forall_n\ \hat{\sigma}^n \in \Psi(\sigma^n),$$

but

$$\hat{\sigma} \not\in \Psi(\sigma) \ \Rightarrow\ \hat{\sigma}_i \not\in \Psi_i(\sigma).$$

Thus,

$$\exists_{\epsilon>0}\, \exists_{\sigma_i'\in\Sigma_i}\; u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon.$$

Since $u_i =$ continuous, there is a sufficiently large N such that

$$\begin{aligned} u_i(\sigma_i', \sigma_{-i}^N) &> u_i(\sigma_i', \sigma_{-i}) - \epsilon \\ &> u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \\ &> u_i(\hat{\sigma}_i^N, \sigma_{-i}^N) + \epsilon. \end{aligned}$$

Thus, $\hat{\sigma}_i^N \notin \Psi(\sigma^N)$, a contradiction.

Thus we conclude that $\Psi : \Sigma \to 2^{\Sigma}$ is a convex valued, uhc map from a nonempty, compact, convex subset Σ of finite dimensional TVS to nonempty subsets of Σ. Thus by Kakutani's fixed point theorem

$$\exists_{\sigma^*\in\Sigma}\; \sigma^* \in \Psi(\sigma^*),$$

and σ^* is a mixed strategy Nash equilibrium.