Beyond Nash: Domination, Rationalization and Correlation
LECTURE 5

Beyond Nash

We have seen that it is impossible to "learn" a Nash equilibrium if we insist on DRIP conditions. A resolution to this dilemma can involve one or more of the following approaches:

1. Explore simpler requirements than Nash equilibria: e.g., undominated sets, rationalizable sets and correlated equilibria. (The first two correspond to minmax and maxmin requirements. The last one requires some side information and may make the system informationally dependent.)

2. Requirement of predictivity may need to be abandoned.

3. Requirement of rationality may need to be abandoned.

Correlated Equilibrium

This concept extends the Nash concept by supposing that the players can build a "correlated device" that sends each of the players a private signal before they choose their strategy.

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Main Ingredients: *Predictions* using only the assumption that the structure of the game (i.e., the strategy spaces and payoffs, S_i's and u_i's) and the rationality of the players are common knowledge.

Iterated Strict Dominance and Rationalizability

Definition **Iterated Strict Dominance**: *Let*

$$S_i^0 = S_i \quad and \quad \Sigma_i^0 = \Sigma_i$$

Let for all $n > 0$

$$S_i^n = \left\{ s_i \in S_i^{n-1} \mid \forall_{\sigma_i' \in \Sigma_i^{n-1}} \exists_{s_{-i} \in S_{-i}^{n-1}} u_i(s_i, s_{-i}) \geq u_i(\sigma_i', s_{-i}) \right\},$$

(Thus s_i dominates all the mixed strategies for some strategy profile of the rivals) and define

$$\Sigma_i^n = \left\{ \sigma_i \in \Sigma_i \mid \sigma_i(s_i) > 0 \Rightarrow s_i \in S_i^n \right\}.$$

Let

$$S_i^\infty = \bigcap_{n=0}^{\infty} S_i^n$$

be the set of player i's pure strategies that survive iterated deletion of strictly dominated strategies.

Let

$$\Sigma_i^\infty = \left\{ \sigma_i = \text{ mixed strategy } \mid \forall_{\sigma_i' \in \Sigma_i} \exists_{s_{-i} \in S_{-i}^\infty} u_i(\sigma_i, s_{-i}) \geq u_i(\sigma_i', s_{-i}) \right\}$$

be the set of player i's mixed strategies that survive iterated deletion of strictly dominated strategies.

Example:

	L	R
U	1,3	-2,0
M	-2,0	1,3
D	0,1	0,1

Note that

$$S_r^0 = \{U, M, D\} \quad \& \quad \Sigma_r^0 = \{\sigma \text{ (with full support) }\}.$$

Similarly,

$$S_c^0 = \{L, R\} \quad \& \quad \Sigma_c^0 = \{\sigma \text{ (with full support) }\}.$$

Also note that

$$S_r^\infty = \cdots = S_r^2 = S_r^1 = S_r^0, \quad \& \quad S_c^\infty = \cdots = S_c^2 = S_c^1 = S_c^0.$$

Note, however, that for all values $p \in (1/3, 2/3)$ the mixed strategy $\sigma_r = (p, 1-p, 0)$ is dominated by D. Thus,

$$\Sigma_r^\infty \subset \Sigma_r^0.$$

Some Properties of Undominated Sets

$$S^\infty = S_1^\infty \times S_2^\infty \times \cdots \times S_I^\infty, \quad \& \quad \Sigma^\infty = \Sigma_1^\infty \times \Sigma_2^\infty \times \cdots \times \Sigma_I^\infty.$$

1. The final surviving strategy spaces are *independent of the elimination order*.

2. A strategy is strictly dominated against all pure strategies of the rivals if and only if it is dominated against all of their

strategies. Thus, the following is an equivalent definition of the undominated sets:

$$S_i^0 = S_i \quad \text{and} \quad \Sigma_i^0 = \Sigma_i$$

$$\begin{aligned} S_i^n &= \Big\{ s_i \in S_i^{n-1} \mid \\ &\qquad \forall_{\sigma_i' \in \Sigma_i^{n-1}} \exists_{s_{-i} \in S_{-i}^{n-1}} \ u_i(s_i, s_{-i}) \geq u_i(\sigma_i', s_{-i}) \Big\}. \\ \Sigma_i^n &= \Big\{ \sigma_i \in \Sigma_i^{n-1} \mid \\ &\qquad \forall_{\sigma_i' \in \Sigma_i^{n-1}} \exists_{s_{-i} \in S_{-i}^{n-1}} \ u_i(\sigma_i, s_{-i}) \geq u_i(\sigma_i', s_{-i}) \Big\}. \end{aligned}$$

$$S_i^\infty = \bigcap_{n=0}^{\infty} S_i^n, \quad \& \quad \Sigma_i^\infty = \bigcap_{n=0}^{\infty} \Sigma_i^n.$$

Definition *A game is solvable by iterated (strict) dominance, if for each player i, S_i^∞ is a singleton, i.e., $S_i^\infty = \{s_i^*\}$. In this case, the strategy profile $(s_1^*, s_2^*, \ldots, s_I^*)$ is a (unique) Nash equilibrium.*

Proof: Suppose that it is not a Nash equilibrium: That is for some i

$$s_i^* \notin BR_i(s_{-i}^*)$$

Thus

$$\exists_{s_i \in S_i} \ u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*).$$

But suppose s_i was eliminated in round n: Then

$$\exists_{s_i' \in S_i^{n-1}} \forall_{s_{-i} \in S_{-i}^{n-1}} \ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

Since $s_{-i}^* \in S_{-i}^\infty$, we have $u_i(s_i', s_{-i}^*) > u_i(s_i, s_{-i}^*)$. Repeating in this fashion we get a sequence of inequalities:

$$u_i(s_i^*, s_{-i}^*) > \cdots > u_i(s_i'', s_{-i}^*) > u_i(s_i', s_{-i}^*) > u_i(s_i, s_{-i}^*),$$

resulting in a contradiction.

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Rationalizability

This notion is due to Bernheim(1984), Pearce(1984) and Aumann(1987) and provides a complementary approach to iterated strict dominance. This approach tries to answer the following question:

> "*What are all the strategies that a rational player can play?* "

Rational player will only play those strategies that are best responses to some beliefs he has about his rivals' strategies.

Definition **(Rationalizable Strategies)** *Let*

$$\tilde{\Sigma}_i^0 = \Sigma_i.$$

For $n > 0$, let

$$\tilde{\Sigma}_i^n = \Big\{ \sigma_i \in \tilde{\Sigma}_i^{n-1} \mid \exists_{\sigma_{-i} \in \times_{j \neq i} Conv(\tilde{\Sigma}_j^{n-1})} \forall_{\sigma_i' \in \tilde{\Sigma}_i^{n-1}} \; u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \Big\}.$$

The rationalizable strategies for player i are

$$R_i = \bigcap_{n=0}^{\infty} \tilde{\Sigma}_i^n$$

A strategy profile σ is rationalizable if σ_i is rationalizable for each player i. Let $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_I^*)$ be a Nash equilibrium. Note first, $\sigma_i^* \in \tilde{\Sigma}_i^0$, for all i. Next assume that $\sigma^* \in \times_i \tilde{\Sigma}_i^{n-1}$. Thus $\sigma_i^* \in \tilde{\Sigma}_i^{n-1}$, and $\sigma_{-i}^* \in \times_{j \neq i} \tilde{\Sigma}_j^{n-1}$. Hence,

$$\forall_{\sigma_i' \in \Sigma_i} \; u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \Rightarrow \quad \sigma_i^* \in \tilde{\Sigma}_i^n.$$

Thus, $\sigma^* \in R = \times_i R_i$.

Hence,

Theorem *Every Nash equilibrium is rationalizable.*