Game Theory & Learning: LECTURE 8

Adaptive and Sophisticated Learning(CONT'D)

Definition *A strategy $x_n \in S_n$ is ϵ-dominated by another strategy $\bar{x}_n \in \Delta(S_n)$ if*

$$\forall_{z_{-n} \in S_{-n}} \ \pi_n(x_n, z_{-n}) + \epsilon < \pi_n(\bar{x}_n, z_{-n}).$$

If $\forall_\epsilon$ x_n is ϵ-dominated by $\bar{x}_n$, then x_n is dominated by $\bar{x}_n$ (in the classical sense).

Let $T \subseteq S$. Define $T_n \equiv T|_{S_n}$ = projection of T onto S_n. $T_{-n} = \times_{j \neq n} T_j$.

Definition *Given $T \subseteq S$. Let*

$$\begin{aligned} U_n^\epsilon(T) &= \{x_n \in S_n : \forall_{y_n \in \Delta(S_n)} \ \exists_{z_{-n} \in T_{-n}} \\ &\qquad \pi_n(x_n, z_{-n}) + \epsilon \geq \pi_n(y_n, z_{-n})\} \\ U^\epsilon(T) &= \times_{n \in N} U_n^\epsilon(T). \end{aligned}$$

$U_n^\epsilon(T)$ = Pure strategies in S_n that are not ϵ-dominated when n's rivals are limited to T_{-n}.

Fact 1
The operator U^ϵ is monotonic. Let R and T be sets of strategy profiles.

$$R \subseteq T \quad \Rightarrow \quad U^\epsilon(R) \subseteq U^\epsilon(T).$$

Fact 2

$$\exists_T \ U^\epsilon(T) \not\subset T.$$

In general, starting with some arbitrary set of strategy profile T one may not be able to create a monotonically descending chain of sets of strategy profiles:

$$T \supseteq U^\epsilon(T) \supseteq U^{\epsilon,2}(T) \supseteq \cdots \supseteq U^{\epsilon,k}(T) \supseteq U^{\epsilon,k+1}(T) \supseteq \cdots$$

Fact 3
However, $S \supseteq U^\epsilon(S)$. Since S is the whole nothing new can get introduced.

By the monotonicity of U^ϵ, we see that if

$$U^{\epsilon,k}(T) \supseteq U^{\epsilon,k+1}(T),$$

then

$$U^{\epsilon}(U^{\epsilon,k}(T)) \supseteq U^{\epsilon}(U^{\epsilon,k+1}(T))$$

and

$$U^{\epsilon,k+1}(T) \supseteq U^{\epsilon,k+2}(T).$$

Putting it all together, we *do* have

$$S \supseteq U^{\epsilon}(S) \supseteq U^{\epsilon,2}(S) \supseteq \cdots \supseteq U^{\epsilon,k}(S) \supseteq U^{\epsilon,k+1}(S) \supseteq \cdots$$

We then define

$$U^{\epsilon\infty}(S) = \bigcap_{k=0}^{\infty} U^{\epsilon,k}(S).$$

Hence, $U^{0\infty}(S) = \lim_{\epsilon\to 0}\ U^{\epsilon,\infty}(S) =$ Serially undominated strategy set. We say x is serially undominated, if $x \in U^{0\infty}(S)$.

Definition *A sequence of strategies $\{x_n(t)\}$ is consistent with adaptive learning by player n if*

$$\forall_{\epsilon>0}\ \forall_{\hat{t}}\ \exists_{\overline{t}}\ \forall_{t\geq\overline{t}}\ x_n(t) \in U_n^{\epsilon}\Big(\{x(s) : \hat{t} \leq s < t\}\Big).$$

A sequence of strategy profiles $\{x(t)\}$ is consistent with adaptive learning if each $\{x_n(t)\}$ has this property.

Looking Forward

$$F^{\epsilon,0}(\hat{t},t) = U^{\epsilon}\Big(\{x(s) : \hat{t} \leq s < t\}\Big).$$

$\forall\ k \geq 1$:

$$F^{\epsilon,k}(\hat{t},t) = U^{\epsilon}\Big(F^{\epsilon,k-1}(\hat{t},t)\ \cup\ \{x(s) : \hat{t} \leq s < t\}\Big).$$

Lemma

$$F^{\epsilon,0}(\hat{t},t) \subseteq F^{\epsilon,1}(\hat{t},t) \subseteq \cdots \subseteq F^{\epsilon,k}(\hat{t},t) \subseteq F^{\epsilon,k+1}(\hat{t},t) \subseteq \cdots$$

Proof

By the monotonicity of U^ϵ,

$$F^{\epsilon,0}(\hat{t},t) \subseteq F^{\epsilon,1}(\hat{t},t).$$

Assume by inductive hypothesis,

$$F^{\epsilon,k-1}(\hat{t},t) \subseteq F^{\epsilon,k}(\hat{t},t).$$

Then

$$\begin{aligned} & F^{\epsilon,k-1}(\hat{t},t) \ \cup \ \{x(s) : \hat{t} \leq s < t\} \\ \subseteq \quad & F^{\epsilon,k}(\hat{t},t) \ \cup \ \{x(s) : \hat{t} \leq s < t\}. \end{aligned}$$

By the monotonicity of U^ϵ,

$$\begin{aligned} & U^\epsilon\Big(F^{\epsilon,k-1}(\hat{t},t) \ \cup \ \{x(s) : \hat{t} \leq s < t\}\Big) \\ \subseteq \quad & U^\epsilon\Big(F^{\epsilon,k}(\hat{t},t) \ \cup \ \{x(s) : \hat{t} \leq s < t\}\Big). \end{aligned}$$

Thus

$$F^{\epsilon,k}(\hat{t},t) \subseteq F^{\epsilon,k+1}(\hat{t},t).$$

Definition *A sequence of strategies $\{x_n(t)\}$ is consistent with sophisticated learning by player n if*

$$\forall_{\epsilon>0}\ \forall_{\hat{t}}\ \exists_{\bar{t}}\ \forall_{t\geq\bar{t}}\ x_n(t) \in U_n^\epsilon(F^{\epsilon\infty}(\hat{t},t)).$$

A sequence of strategy profiles $\{x(t)\}$ is consistent with sophisticated learning if each $\{x_n(t)\}$ has this property.

$$\forall_{\epsilon>0}\ \forall_{\hat{t}}\ \exists_{\bar{t}}\ \forall_{t\geq\bar{t}}\ x(t) \in F^{\epsilon\infty}(\hat{t},t).$$