Game Theory & Learning: LECTURE12

Information Theory and Learning (CONT'D)

Gambling and Entropy

Horse Race

$$\#\text{ horses } = m, \quad \{H_1, H_2, \ldots, H_m\}.$$

$$p_i = Pr[H_i \text{ wins }]$$

$$u_i = \text{ pay-off if } H_i \text{ wins }.$$

If $b_i =$ bet on the ith horse then the payoff $=$

$$\begin{cases} b_i u_i, & \text{if } H_i \text{ wins with probability } p_i; \\ 0, & \text{if } H_i \text{ loses with probability } (1-p_i). \end{cases}$$

Assume that the gambler has 1 dollar. Let $b_i =$ fraction of his wealth invested in H_i. Thus

$$0 \leq b_i \leq 1. \qquad \sum_{i=1}^{m} b_i = 1.$$

Note that the gambler's pay-off is $b_i u_i$ if H_i wins (with probability p_i.)

$$S(X) = b(X)u(X)$$

= factor by which the gambler increases his wealth if X wins.

Repeated game with reinvestment.

$$\begin{aligned} S_0 &= 1, \\ S_n &= S_{n-1}S(X_n), \qquad \text{if } X_n \text{ wins in the } n\text{th game.} \end{aligned}$$

Thus

$$S_n = \prod_{i=1}^{n} S(X_i) = 2^{\sum \lg S(X_i)}.$$

Let

$$\mathbb{E}_p[\lg S(X)] = \sum p_k \lg(b_k u_k) = W(b, p) = \text{ Doubling Rate,}$$

where $b =$ the betting strategy. Then

$$\frac{1}{n} \lg S_n \to \mathbb{E}_p[\lg S(X)] \qquad \text{in probability,}$$

by "Law of Large Number." Hence

$$S_n \to 2^{nW(b,p)}.$$

Definition **Doubling Rate**

$$W(b, p) = \sum_{k=1}^{m} p_k \lg(b_k u_k).$$

Theorem *Let the race outcomes $X_1, \ldots, X_n$ be i.i.d.$\sim p(x)$. Then the wealth of the gambler using betting strategy b grows exponentially at rate $W(b, p)$, i.e.*

$$S_n \equiv 2^{nW(b,p)}.$$

$$\begin{aligned} W(b,p) &= \sum p_k \lg(b_k u_k) \\ &= \sum p_k \left[\lg \frac{b_k}{p_k} - \lg \frac{1}{p_k} + \lg u_k\right] \\ &= \sum p_k \lg u_k - H(p) - D(p\|b) \\ &\leq \sum p_k \lg u_k - H(p), \end{aligned}$$

with equality iff $p = b$.

The optimal doubling rate

$$W^*(p) = \max_b W(b,p) = W(p,p) = \sum p_k \lg u_k - H(p).$$

Theorem Proportional gambling is log-optimal.
The optimum doubling rate is given by

$$W^*(p) = W(b^*,p) = \sum p_k \lg u_k - H(p),$$

and is achieved by the proportional gambling scheme, $b^* = p$.

Define $r_k = \frac{1}{u_k}$ = Bookie's estimate of the win "probabilities." Thus

$$\sum_k r_k = \sum \frac{1}{u_k} = 1.$$

Odds are fair and there is no track take.

$$\begin{aligned} W(b,p) &= \sum p_k \lg \frac{b_k}{r_k} \\ &= \sum p_k \left[\lg \frac{b_k}{p_k} - \lg \frac{r_k}{p_k}\right] \\ &= D(p\|r) - D(p\|b). \end{aligned}$$

Doubling Rate = Difference between the distance of the bookie's estimate from the true distribution and the distance of the gambler's estimate from the true distribution.

Special Case: Odds are m-for-1 on each horse:

$$\forall_k \ \ r_k = \frac{1}{m}.$$

Thus,

$$W^*(p) \ = \ D(p||u) - D(p||b^*) \ = \ \lg m - H(p).$$

Theorem **Conservation Theorem**

$$W^*(p) + H(p) = \lg m$$

for uniform odds.

Low-Entropy Races are Most Profitable.
Case of a not fully invested gambler.

$$\begin{aligned} b_0 &= \text{wealth held out as cash} \\ b_i &= \text{proportional bet on } H_i. \end{aligned}$$

$$b_0 \geq 0, \quad b_i \geq 0, \quad \sum_{i=0}^{m} b_i = 1.$$

Thus

$$S(X) = b_0 + b(X)u(X).$$

- **Fair Odds:** $\sum \frac{1}{u_i} = 1$.
  If there is a non-fully-invested strategy with $b_0, b_1, \ldots, b_m$, then there is also a full investment as follows

$$\begin{aligned} b'_0 &= 0 \\ b'_i &= b_i + \frac{b_0}{u_i}, \quad 1 \leq i \leq m \\ \sum_{i=0}^{m} b'_i &= \sum_{i=1}^{m} b_i + b_0 \sum_{i=1}^{m} \frac{1}{u_i} = 1. \end{aligned}$$

  Thus

$$\begin{aligned} S(X) &= b'(X)u(X) = \frac{b_0}{u(X)}u(X) + b(X)u(X) \\ &= b_0 + b(X)u(X). \end{aligned}$$

  Thus in this case there is a risk-neutral investment.

- **Super-Fair Odds:** $\sum \frac{1}{u_i} < 1$.
  "Dutch Book" betting strategy.

$$b_0 = 1 - \sum \frac{1}{u_i}, \quad b_i = \frac{1}{u_i}, 1 \leq i \leq m.$$

Thus

$$S(X) = \left(1 - \sum \frac{1}{u_i}\right) + \frac{1}{u(X)} u(X) = 2 - \sum \frac{1}{u_i} > 1$$

with no risk!

This, however, implies a strong arbitrage opportunity.

- **Sub-Fair Odds:** $\sum \frac{1}{u_i} > 1$.
  In this case, proportional gambling is no longer log-optimal and this case represents a risky undertaking for the gambler.

Side Information

Some external information about the performance of the horses may be available—for instance, previous games.
$X = \{1, 2, \ldots, m\}$, represent the horses.
$Y =$ Some other arbitrary discrete random variable (*Side Information*).

$$p(x, y) = \text{ joint probability mass function for } (X, Y) \ .$$

$b(x|y) =$ conditional betting strategy depending on Y
$=$ proportion of wealth bet on horse x given that $y \in Y$ is observed.
$b(x) =$ unconditional betting strategy.

$$\begin{array}{ll} b(x) \geq 0, & \sum_x b(x) = 1. \\ b(x|y) \geq 0, & \sum_x b(x|y) = 1. \end{array}$$

$$\begin{aligned} W^*(X) &= \max_{b(x)} \sum_x p(x) \lg(b(x)u(x)) \\ &= \sum_x p(x) \lg u(x) - H(X). \end{aligned}$$

$$\begin{aligned} W^*(X|Y) &= \max_{b(x|y)} \sum_x p(x) \lg(b(x|y)u(x)) \\ &= \sum_x p(x) \lg u(x) - H(X|Y). \end{aligned}$$

$$\begin{aligned} \Delta W &= W^*(X|Y) - W^*(X) \\ &= \sum_x p(x) \lg u(x) - H(X|Y) - \sum_x p(x) \lg u(x) + H(X) \\ &= H(X) - H(X|Y) = I(X;Y) \geq 0. \end{aligned}$$

Increase in Doubling Rate =
Mutual information between the horse race and side information.

Learning

$\{X_k\}$ = Sequence of horse race outcomes from a stochastic process.

$$\begin{aligned} & W^*(X_k|X_{k-1}, X_{k-2}, \ldots, X_1) \\ &= \mathbb{E}\left[\max_{b(.|x_{k-1},\ldots,x_1)} E[\lg S(X_k)|X_{k-1}, X_{k-2}, \ldots, X_1]\right] \\ &= \lg m - H(X_k|X_{k-1}, X_{k-2}, \ldots, X_1), \end{aligned}$$

and is maximized for

$$b^*(x_k|x_{k-1}, \ldots, x_1) = p(x_k|x_{k-1}, \ldots, x_1).$$

Note that since

$$S_n = \prod_{i=1}^{n} S(X_i),$$

we have

$$\begin{aligned}
\frac{1}{n} E \lg S_n &= \frac{1}{n} \sum E \lg S(X_i) \\
&= \frac{1}{n} \sum \left(\lg m - H(X_i | X_1, \ldots, X_{i-1}) \right) \\
&= \lg m - \frac{H(X_1, \ldots, X_n)}{n} \\
&= \lg m - H(\mathcal{X}).
\end{aligned}$$

$H(\mathcal{X})$ is simply the entropy rate.