Universal Portfolio

Universal Portfolio

1. Sequential Portfolio Selection Procedure. An adapted process.
2. No statistical assumption about the behavior of the market.
3. Robust procedure with respect to arbitrary market sequences occurring in the real world.

We shall consider growth of wealth for arbitrary market sequences. For example, our goal may be to outperform the best buy-and-hold strategy—i.e., we wish to be competitive against a competing investor who can predict n future days. A different goal may be to outperform all constant rebalanced portfolio strategies.

$$\begin{aligned} m &= \#\text{ stocks traded in a market} \\ x_i &= \text{price relative for the } i\text{th stock} \\ &= \frac{\text{stock price at close}}{\text{stock price at open}} = \frac{P_i(c)}{P_i(o)} \\ &= 1 + \frac{\Delta P_i}{P_i}. \end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \text{ stock market vector .}$$

Portfolio

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \text{ portfolio ,} \quad \begin{cases} b_i \geq 0 \\ \sum_i b_i = 1. \end{cases}$$

Portfolio is simply the *proportion of the current wealth invested in each of the stocks.*

$$S = b \cdot x = b^T x = \sum_i b_i x_i,$$

= Factor by which the wealth increases in one period.

$$x(1), x(2), \ldots, x(n)$$

= stock market vectors for n consecutive days.

$$b = \text{ Fixed (constant) portfolio}$$

We shall follow a constant rebalanced portfolio strategy.

$$S_n(b) = \prod_{i=1}^{n} b^T x(i), \quad \begin{cases} S_0(b) &= 1 \\ S_n(b) &= S_{n-1}(b)\ b^T x(n). \end{cases}$$

$$S_n^* = \max_b S_n(b) = S_n(b^*).$$

This is the maximum wealth achievable on the given stock sequence maximized over all constant rebalanced portfolios.

Universal Portfolio Strategy

$$\hat{b}(k)$$

depends only the past price relatives: $x(1)$, $x(2)$, $\ldots$, $x(k-1)$.

It performs as well as the best constant rebalanced portfolio based on a clairvoyant knowledge of the sequence of price relatives.

Questions

Since we wish to compete against a clairvoyant investor (who knows the future) and universal portfolios only depend on the past (past has no *causal* or *correlated* relation with the future), how is it possible that universal portfolio can be competitive?

Malicious/adversarial nature is free to structure the future so as to help the competing investor.

$$\hat{b}(1) = \begin{pmatrix} 1/m \\ 1/m \\ \vdots \\ 1/m \end{pmatrix}.$$

$$S_k(b) = \prod_{i=1}^{k} b^T x(i), \quad B = \left\{ b \in \mathbb{R}_+^m \mid b_i \geq 0, \sum b_i = 1 \right\}.$$

$$\hat{b}(k+1) = \frac{\int_B b S_k(b) db}{\int_B S_k(b) db}$$

Note that

$$\hat{b}(k+1)^T x(k+1) = \frac{\int_B b^T x(k+1) S_k(b)\, db}{\int_B S_k(b) db} = \frac{\int_B S_{k+1}(b) db}{\int_B S_k(b) db}$$

The "learned" portfolio is the performance weighted average of all portfolios $b \in B$.

Thus

$$\hat{S}_n = \prod_{k=1}^{n} \hat{b}(k)^T x(k) = \frac{\int_B S_n(b)db}{\int_B db} = (m-1)! \int_B S_n(b)db.$$

We will show that

$$\hat{S}_n \approx S_n^* \frac{(m-1)!(\sqrt{2\pi/n})^{m-1}}{\sqrt{|J_n|},}$$

where $J_n =$ a positive semidefinite $(m-1) \times (m-1)$ sensitivity matrix.

Properties & Analysis

Let F be some arbitrary probability distribution for price relatives over $\mathbb{R}_+^m$. Let F_n be the empirical distribution associated with $x(1)$, $x(2)$, ..., $x(n)$. $Pr[X = x(i)] = 1/n$. $Pr[X \neq x(i), \forall_i] = 0$.

$$\lim_{n \to infty} F_n \to F.$$

Doubling Ratio

$$\begin{aligned} W(b,F) &= \int \lg(b^T x) dF(x) \\ W(b,F_n) &= \sum_{i=1}^{n} \frac{1}{n} \lg(b^T x(i)) \\ W^*(F) &= \max_b W(b,F) \\ W^*(F_n) &= \max_b W(b,F_n) \end{aligned}$$

$$S_n^* = \max_b S_n(b) = \max_b \prod_{i=1}^{n} b^T x(i) = 2^{nW^*(F_n)}.$$

Let e_j be the vector

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad 1 \text{ in } j\text{th position only.}$$

$$\begin{aligned} S_n(e_j) &= \prod_{k=1}^{n} e_j^T x(k) = \prod_{k=1}^{n} x_j(k) \\ &= \text{Wealth due to buy-and-hold strategy} \\ &\quad\ \text{associated wit the } j\text{th stock.} \end{aligned}$$

Since S_n^* is a maximization of $S_n(b)$ over the entire simplex,

$$\forall_j \ \ S_n^* \geq S_n(e_j).$$

Corollary

1. **Target Exceeds Best Stock.**

$$S_n^* \geq \max_j S_n(e_j).$$

2. **Target Exceeds Value Line.**

$$S_n^* \geq \left(\prod_j S_n(e_j) \right)^{1/m}$$

3. **Target Exceeds Arithmetic Mean.**

$$S_n^* \geq \sum_j \alpha_j S_n(e_j), \qquad \alpha_j \geq 0, \sum_j \alpha_j = 1.$$

4. $S_n^*(x(1),\ x(2),\ \ldots,\ x(n))$ is invariant under permutations of the sequence $x(1),\ x(2),\ \ldots,\ x(n)$.

Lemma

$$\hat{S}_n = \prod_{k=1}^{n} \hat{b}(k)^T x(k) = \frac{\int_B S_n(b)db}{\int_B db}$$

where

$$S_n(b) = \prod_{i=1}^{n} b^T x(i).$$

$\hat{S}_n$ = *Wealth from universal portfolio is the average of* $S_n(b)$ *over the simplex.*

Proof*:*

Recall that

$$\hat{b}(k+1)^T x(k+1) = \frac{\int S_{k+1}(b)db}{\int S_k(b)db}.$$

Telescoping the products

$$\begin{aligned}
\hat{S}_n &= \prod_{k=1}^{n} \hat{b}(k)^T x(k) \\
&= \frac{\int S_n(b)db}{\int S_{n-1}(b)db} \times \cdots \times \frac{\int S_1(b)db}{\int db} \\
&= \frac{\int S_n(b)db}{\int db} \\
&= \frac{\int_B \prod_{i=1}^{n} b^T x(i)db}{\int_B db} \\
&= \mathbb{E}_b S_n(b) = \mathbb{E}_b 2^{nW(b,F_n)}.
\end{aligned}$$

Corollary $\hat{S}_n(x(1),\ x(2),\ \ldots,\ x(n))$ *is invariant under permutations of the sequence* $x(1),\ x(2),\ \ldots,\ x(n)$.

Claim

$$\mathbb{E}_b W(b, F_n) \geq \frac{1}{m} \sum_j W(e_j, F_n).$$

$$\mathbb{E}_b W(b, F_n) \ = \ \mathbb{E}_b \int \lg(b^T x)\ dF_n(x)$$

$$\begin{aligned}
&= \mathbb{E}_b \int \lg \sum b_j (e_j^T x)\ dF_n(x) \\
&\geq \mathbb{E}_b \sum b_j \int \lg(e_j^T x)\ dF_n(x) \\
&= \frac{1}{m} \sum \int \lg(e_j^T x) dF_n(x) \\
&= \frac{1}{m} \sum_j W(e_j, F_n).
\end{aligned}$$

By Jensen's inequality

$$\begin{aligned}
E_b 2^{nW(b,F_n)} &\geq 2^{nE_b W(b,F_n)} \\
&\geq 2^{1/m \sum nW(e_j,F_n)} \\
&\geq \left(\prod 2^{nW(e_j,F_n)}\right)^{1/m}.
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{S}_n &= \mathbb{E}_b S_n(b) = \mathbb{E}_b 2^{nW(b,F_n)} \\
&\geq \left(\prod 2^{nW(e_j,F_n)}\right)^{1/m} \geq \left(\prod_{j=1}^{m} S_n(e_j)\right)^{1/m}.
\end{aligned}$$

Corollary *Universal portfolio exceeds Value Line index.*

$$\hat{S}_n \geq \left(\prod_{j=1}^{m} S_n(e_j)\right)^{1/m}.$$

Competitiveness

$F_n(x)$ = Empirical probability mass function. Mass on each $x(i) \in \mathbb{R}_+^m$ is $\frac{1}{n}$.

$$\begin{aligned}
S_n(b) &= \prod_{i=1}^{n} b^T x(i) = 2^{nW(b,F_n)} = e^{nV(b,F_n)}, \\
b^*(F_n) = b^* &= \arg\max S_n(b) = \arg\max V(b, F_n) \in \mathbb{R}_+^m.
\end{aligned}$$

$$S_n^* = \max_{b \in B} S_n(b) = e^{nV^*(F_n)}.$$

Definition *All stocks are active at time n, if*

$$\exists_{b^*; S_n(b^*)=S_n^*} \forall_{i\in[1..m]} \left(b^*(F_n)\right)_i > 0.$$

All stocks are strictly active at time n, if

$$\forall_{b^*; S_n(b^*)=S_n^*} \forall_{i\in[1..m]} \left(b^*(F_n)\right)_i > 0.$$

If

$$\text{Lin } (x(1), x(2), \ldots, x(n)) = \mathbb{R}^m,$$

then we say that the price relatives $x(1)$, $x(2)$, ..., $x(n)$ are of full rank.

$$J(b) = (m-1) \times (m-1) \text{ matrix }.$$

$J(b) =$ Sensitivity Matrix Function of a market with respect to distribution $F(x)$, $x \in \mathbb{R}_+^m$.

$$J_{ij}(b) = \int \frac{(x(i) - x(m))(x(j) - x(m))}{(b^T x)^2} dF(x).$$

$J^* = J(b^*) =$ Sensitivity Matrix.

$$J_{ij}^* = -\frac{\partial^2 V\left((b_1^*, \ldots, b_{m-1}^*, 1 - \sum_{i=1}^{m-1} b_i^*), F\right)}{\partial b_i \, \partial b_j}$$

$=$ Positive Semidefinite Matrix.

It is positive definite if all stocks are strictly active.

Let

$$C = \left\{(c_1, c_2, \ldots, c_{m-1}) \mid c_i \geq 0, \sum c_i \leq 1\right\}.$$

Define

$$b(c) = \left(c_1, \ldots, c_{m-1}, 1 - \sum_{i=1}^{m-1} c_i\right).$$

Thus

$$V_n(c) = \frac{1}{n} \sum_{i=1}^{n} \ln\left(b(c)^T x(i)\right) = \int \ln(b^T x) \; dF_n(x) \equiv \mathbb{E}_{F_n} \ln(b^T x).$$

Using Taylor series expansion:

$$\begin{aligned} V_n(c) &= V_n(c^*) + (c-c^*)^T \nabla V_n(c^*) \\ &\quad - \frac{1}{2}(c-c^*)^T J_n^*(c-c^*) \\ &\quad + \frac{1}{6}\sum_{ijk}(c_i - c_i^*)(c_j - c_j^*)(c_k - c_k^*) \\ &\quad \times \mathbb{E}_{F_n} \frac{(x(i)-x(m))(x(j)-x(m))(x(k)-x(m))}{S^3(\tilde{c})} \end{aligned}$$

where

$$\begin{aligned} \tilde{c} &= \lambda c^* + (1-\lambda)c, \quad \lambda \geq 0, \\ S(\tilde{c}) &= \sum_{i=1}^{m-1} b(\tilde{c})_i X(i). \end{aligned}$$

Assume that all stocks are strictly active:

$$J^* = -\left[\frac{\partial^2 V}{\partial c_i \partial c_j}\right] = \text{ positive definite.}$$

Hence its determinant is strictly positive:

$$|J^*| > 0.$$

Let $u = \sqrt{n}(c - c^*)$. Then since the second term is 0 in the Taylor series formulation, we have

$$\begin{aligned} nV_n(c) &= nV^*(F_n) - \frac{1}{2}u^T J_n^* u \\ &\quad + \frac{1}{6\sqrt{n}}\sum u_i u_j u_k \\ &\quad \times \mathbb{E}_{F_n} \frac{(x(i)-x(m))(x(j)-x(m))(x(k)-x(m))}{S^3(\tilde{c})}. \end{aligned}$$

Next assume that $0 < a \leq x(i) \leq c < \infty$.

$$S(\tilde{c}) \geq a, \quad x(i) - x(m) \leq 2c.$$

Thus the last term in the preceding expression can be bounded by

$$-\frac{1}{6\sqrt{n}}\|u\|^3 m^{3/2}\frac{(2c)^3}{a^3}.$$

Hence, we have

$$nV_n(c) \;=\; nV^*(F_n) - \frac{1}{2}u^T J_n^* u - \frac{4m^{3/2}c^3}{3\sqrt{n}a^3}\|u\|^3.$$

We thus conclude that

$$\begin{aligned} S_n(c) = 2^{nW_n(c)} &\geq e^{(nV_n^*)-(u^T J_n^* u/2)-(4m^{3/2}c^3\|u\|^3/3\sqrt{n}a^3)} \\ &= S_n^*\, e^{-(u^T J_n^* u/2)-(4m^{3/2}c^3\|u\|^3/3\sqrt{n}a^3)}. \end{aligned}$$

Since $\hat{S}_n = \int S_n(b)\,db / \int db$, and since $\int db = 1/(m-1)!$, we have

$$\hat{S}_n \geq S_n^*(m-1)!\ \int_{u\in U} e^{-(u^T J_n^* u/2)-(4m^{3/2}c^3\|u\|^3/3\sqrt{n}a^3)} \left(\frac{1}{\sqrt{n}}\right)^{m-1}\ du.$$

Thus,

$$\hat{S}_n \approx S_n^*\,\frac{(m-1)!(2\pi/n)^{(m-1)/2}}{|J_n|^{1/2}}.$$

In other words,

$$\frac{1}{n}\lg\frac{S_n^*}{\hat{S}_n} = \frac{1}{n}\lg\frac{|J_n|^{1/2}}{(m-1)!(2\pi/n)^{(m-1)/2}} \;\to\; 0, \qquad \text{as } n\to\infty\ .$$

Summarizing, we have

$$\begin{aligned} \frac{1}{n}\lg S_n^* &\sim \frac{1}{n}\lg\hat{S}_n \\ V_n^* &\sim \hat{V}_n. \end{aligned}$$