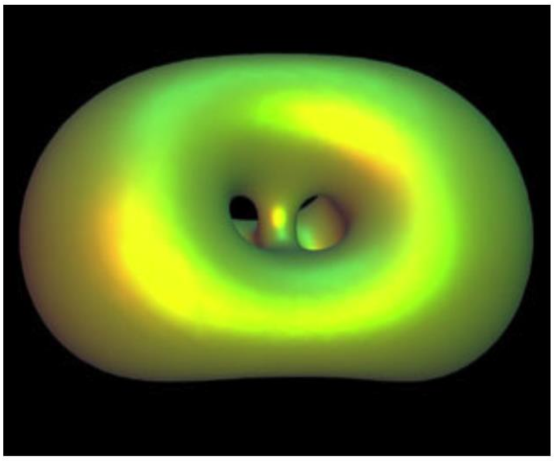
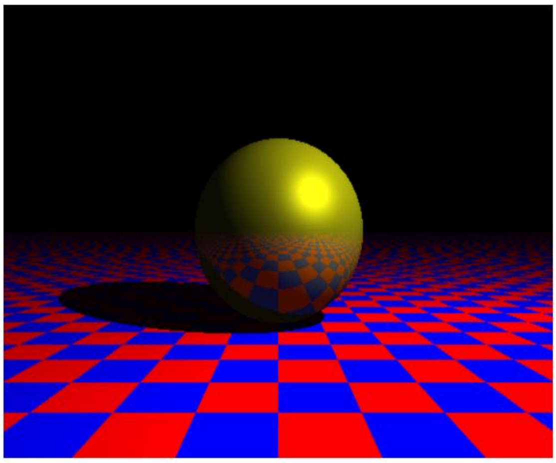
# 1 Introduction

This is a course on surfaces. Your mental image of a surface should be something like this:

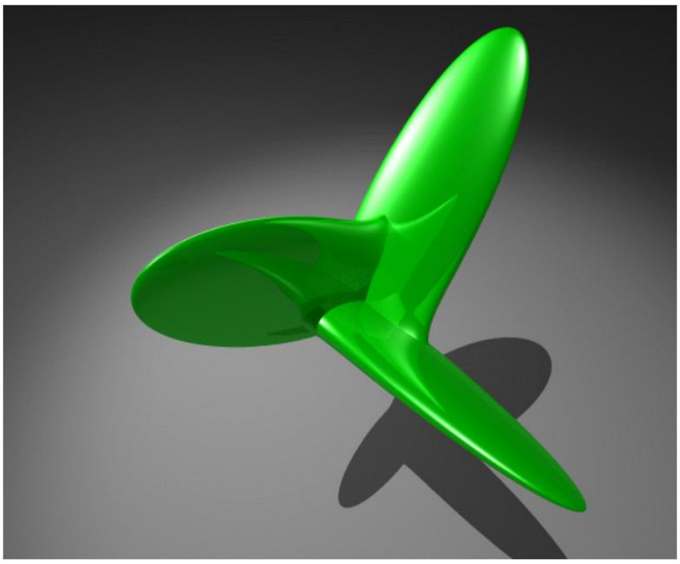


or this



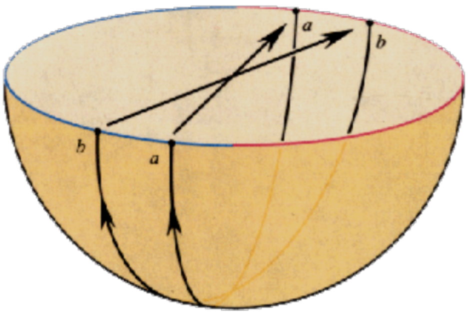
However we are also going to try and consider surfaces intrinsically, or abstractly, and not necessarily embedded in three-dimensional Euclidean space like the two above. In fact lots of them simply can't be embedded, the most notable being the projective plane. This is just the set of lines through a point in $\mathbf{R}^3$ and is as firmly connected with familiar Euclidean geometry as anything. It *is* a surface but it doesn't sit in Euclidean space.

If you insist on looking at it, then it maps to Euclidean space like this



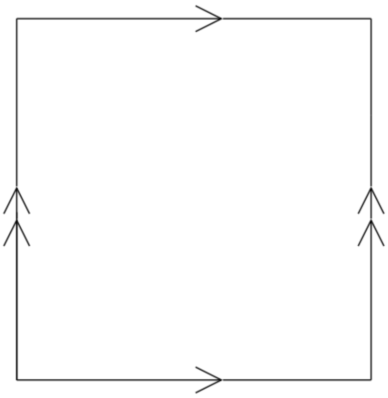
– called *Boy's surface*. This is not one-to-one but it does intersect itself reasonably cleanly.

A better way to think of this space is to note that each line through 0 intersects the unit sphere in two opposite points. So we cut the sphere in half and then just have to identify opposite points on the *equator*:

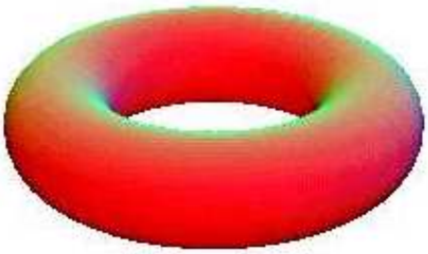



... and this gives you the projective plane.

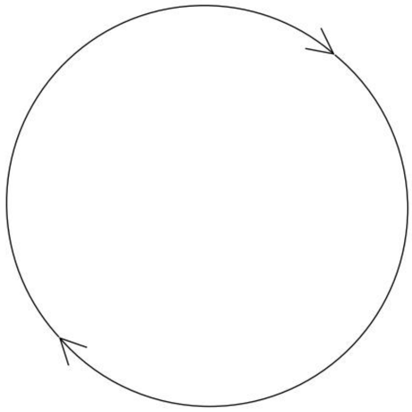
Many other surfaces appear naturally by taking something familiar and performing identifications. A doubly periodic function like $f(x,y) = \sin 2\pi x \cos 2\pi y$ can be thought of as a function on a surface. Since its value at $(x,y)$ is the same as at $(x+m, y+n)$ it is determined by its value on the unit square but since $f(x,0) = f(x,1)$ and $f(0,y) = f(1,y)$ it is really a continuous function on the space got by identifying opposite sides:



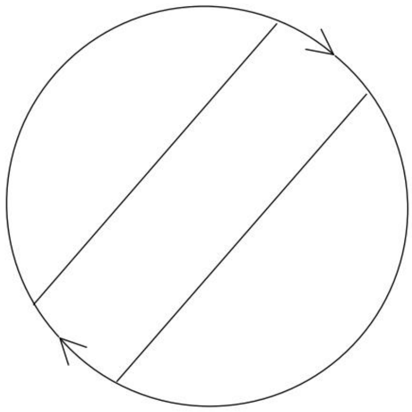
and this is a torus:



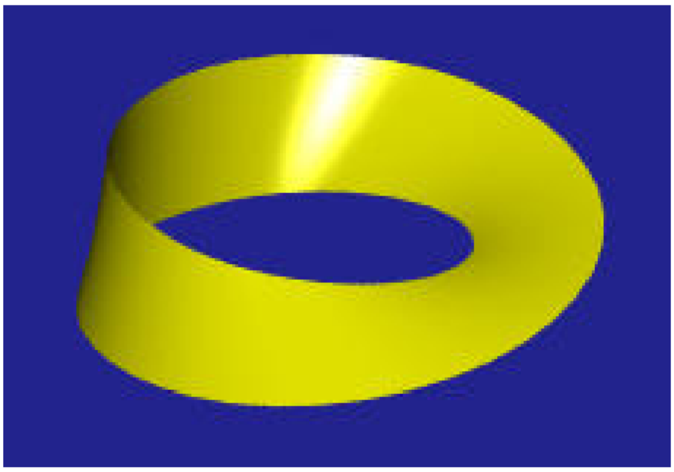
We shall first consider surfaces as topological spaces. The remarkable thing here is that they are completely classified up to homeomorphism. Each surface belongs to two classes – the orientable ones and the non-orientable ones – and within each class there is a non-zero integer which determines the surface. The orientable ones are the ones you see sitting in Euclidean space and the integer is the number of holes. The non-orientable ones are the "one-sided surfaces" – those that contain a Möbius strip – and projective space is just such a surface. If we take the hemisphere above and flatten it to a disc, then projective space is obtained by identifying opposite points on the boundary:



Now cut out a strip:



and the identification on the strip gives the Möbius band:



As for the integer invariant, it is given by the *Euler characteristic* – if we subdivide a surface $A$ into $V$ vertices, $E$ edges and $F$ faces then the Euler characteristic $\chi(A)$ is defined by

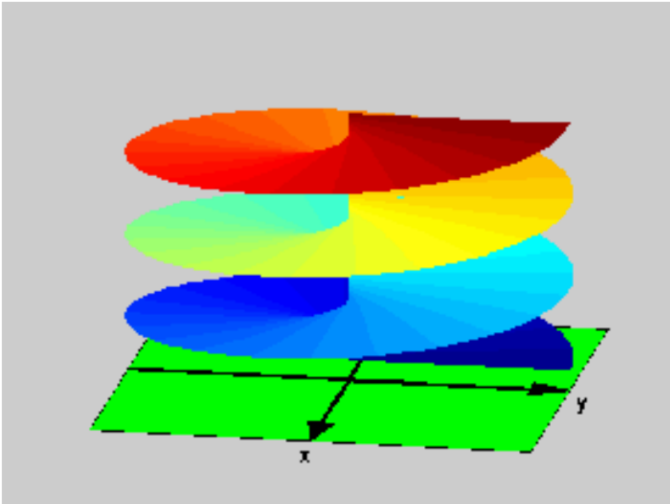
$$\chi(A) = V - E + F.$$

For a surface in Euclidean space with $g$ holes, $\chi(A) = 2 - 2g$. The invariant $\chi$ has the wonderful property, like counting the points in a set, that

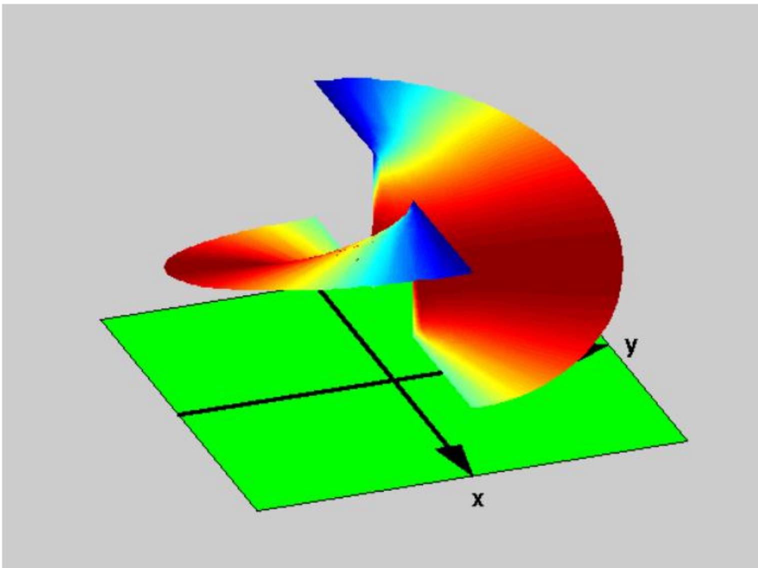
$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

and this means that we can calculate it by cutting up the surface into pieces, and without having to imagine the holes.

One place where the study of surfaces appears is in complex analysis. We know that $\log z$ is not a single valued function – as we continue around the origin it comes back to its original value with $2\pi i$ added on. We can think of $\log z$ as a single valued function on a surface which covers the non-zero complex numbers:




The Euclidean picture above is in this case a reasonable one, using the third coordinate to give the imaginary part of $\log z$: the surface consists of the points $(re^{i\theta}, \theta) \in \mathbf{C} \times \mathbf{R} = \mathbf{R}^3$ and $\log z = \log r + i\theta$ is single-valued. But if you do the same to $\sqrt{z(z-1)}$ you get

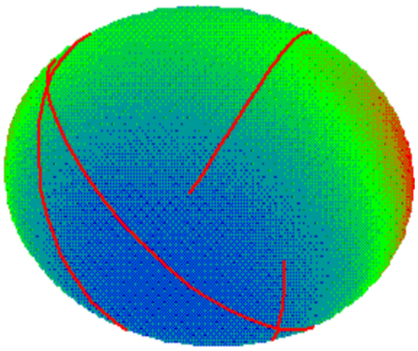


a surface with self-intersections, a picture which is not very helpful. The way out is to leave $\mathbf{R}^3$ behind and construct an abstract surface on which $\sqrt{z(z-1)}$ is single-valued. This is an example of a *Riemann surface*. Riemann surfaces are always orientable, and for $\sqrt{z(z-1)}$ we get a sphere. For $\sqrt{z(z-1)(z-a)}$ it is a torus, which amongst other things is the reason that you can't evaluate

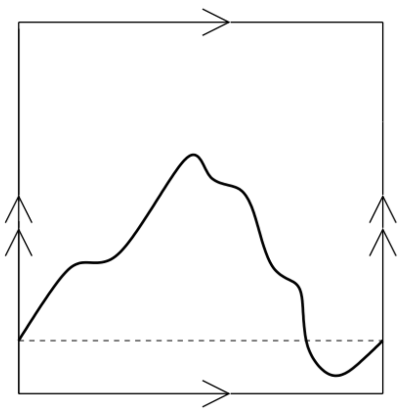
$$\int \frac{dx}{\sqrt{x(x-1)(x-a)}}$$

using elementary functions. In general, given a multi-valued meromorphic function, the Euler characteristic of the Riemann surface on which it is defined can be found by a formula called the Riemann-Hurwitz formula.

We can look at a smooth surface in Euclidean space in many ways – as a topological space as above, or also as a *Riemannian manifold*. By this we mean that, using the Euclidean metric on $\mathbf{R}^3$, we can measure the lengths of curves on the surface.



If our surface is not sitting in Euclidean space we can consider the same idea, which is called a Riemannian metric. For example, if we think of the torus by identifying the sides of a square, then the ordinary length of a curve in the plane can be used to measure the length of a curve on the torus:



A Riemannian metric enables you to do much more than measure lengths of curves: in particular you can define areas, curvature and geodesics. The most important notion of curvature for us is the Gaussian curvature which measures the deviation of formulas for triangles from the Euclidean ones. It allows us to relate the differential geometry of the surface to its topology: we can find the Euler characteristic by integrating the Gauss curvature over the surface. This is called the *Gauss-Bonnet theorem.* There are other analytical ways of getting the Euler characteristic – one is to count the critical points of a differentiable function.

Surfaces with constant Gaussian curvature have a special role to play. If this curvature is zero then locally we are looking at the Euclidean plane, if positive it is the round sphere, but the negative case is the important area of hyperbolic geometry. This has a long history, but we shall consider the concrete model of the upper half-plane as a surface with a Riemannian metric, and show how its geodesics and isometries provide the axiomatic properties of non-Euclidean geometry and also link up with complex analysis. The hyperbolic plane is a surface as concrete as one can imagine, but is an abstract one in the sense that it is not in $\mathbf{R}^3$.

## Four classes of surfaces

Our goal is to study and relate four classes of surfaces:

(1) Topological surfaces (*topological 2-manifolds*),
(2) Smooth surfaces in $\mathbb{R}^3$ (*smooth real 2-manifolds embedded in* $\mathbb{R}^3$),
(3) Abstract smooth surfaces (*smooth real 2-manifolds*),
(4) Riemann surfaces (*complex 1-manifolds*).

We will postpone the precise definition to later. For now, the rough idea is that a surface locally looks like a 2-dimensional disc. Whether it looks like the disc continuously, smoothly or holomorphically distinguishes the cases (1), (2)/(3), and (4) respectively. The reason for studying (2) before (3) is that you already know what it means for functions on $\mathbb{R}^3$ to be smooth (infinitely differentiable), whereas in (3) the definition is a little more difficult because you need to first define local smooth coordinates on the surface. Some surfaces are part of all four classes (such as a torus), others only of some, but all our surfaces belong to class (1).

REFERENCES

1. Boothby, William M. (1986), An introduction to differentiable manifolds and Riemannian geometry, Pure and Applied Mathematics, 120 (2nd ed.), Academic Press, ISBN 0121160521

2. Cartan, Élie (1983), Geometry of Riemannian Spaces, Math Sci Press, ISBN 978-0-915692-34-7; translated from 2nd edition of Leçons sur la géométrie des espaces de Riemann (1951) by James Glazebrook.

3. do Carmo, Manfredo P. (2016), Differential Geometry of Curves and Surfaces (revised & updated 2nd ed.), Mineola, NY: Dover Publications, Inc., ISBN 0-486-80699-5

4. do Carmo, Manfredo (1992), Riemannian geometry, Mathematics: Theory & Applications, Birkhäuser, ISBN 0-8176-3490-8

5. Gray, Alfred; Abbena, Elsa; Salamon, Simon (2006), Modern Differential Geometry of Curves And Surfaces With Mathematica®, Studies in Advanced Mathematics (3rd ed.), Boca Raton, FL: Chapman & Hall/CRC, ISBN 978-1-58488-448-4

6. Toponogov, Victor A. (2005), Differential Geometry of Curves and Surfaces: A Concise Guide, Springer-Verlag, ISBN 978-0-8176-4384-3

7. Valiron, Georges (1986), The Classical Differential Geometry of Curves and Surfaces, Math Sci Press, ISBN 978-0-915692-39-2 Full text of book

8. Warner, Frank W. (1983), Foundations of differentiable manifolds and Lie groups, Graduate Texts in Mathematics, 94, Springer, ISBN 0-387-90894-3

9. Geometry of Surfaces (Universitext) Corrected Edition by John Stillwell

10. Elements of Algebra: Geometry, Numbers, Equations (Undergraduate Texts in mathematics by John Stillwell

11. Differential Geometry of Curves and Surfaces by Thomas F. Banchoff, Stephen T. Lovett

12. Introduction to Topology: Third Edition (Dover Books on Mathematics) by Bert Mendelson