2 The topology of surfaces

2.1 The definition of a surface

We are first going to consider surfaces as topological spaces, so let's recall some basic properties:

Definition 1 *A topological space is a set X together with a collection $\mathcal{T}$ of subsets of X (called the 'open subsets' of X) such that*

- $\emptyset \in \mathcal{T}$ *and* $X \in \mathcal{T}$;
- *if* $U, V \in \mathcal{T}$ *then* $U \cap V \in \mathcal{T}$;
- *if* $U_i \in \mathcal{T} \quad \forall i \in I$ *then* $\bigcup_{i \in I} U_i \in \mathcal{T}$.

- X is called Hausdorff if whenever $x, y \in X$ and $x \neq y$ there are open subsets U, V of X such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.
- A map $f : X \to Y$ between topological spaces X and Y is called continuous if $f^{-1}(V)$ is an open subset of X whenever V is an open subset of Y.
- $f : X \to Y$ is called a homeomorphism if it is a bijection and both $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ are continuous. Then we say that X is homeomorphic to Y.
- X is called compact if every open cover of X has a finite subcover.

Subsets of $\mathbf{R}^n$ are Hausdorff topological spaces where the open sets are just the intersections with open sets in $\mathbf{R}^n$. A surface has the property that near any point it looks like Euclidean space – just like the surface of the spherical Earth. More precisely:

Definition 2 *A topological surface (sometimes just called a* surface*) is a Hausdorff topological space X such that each point x of X is contained in an open subset U which is homeomorphic to an open subset V of $\mathbf{R}^2$.*

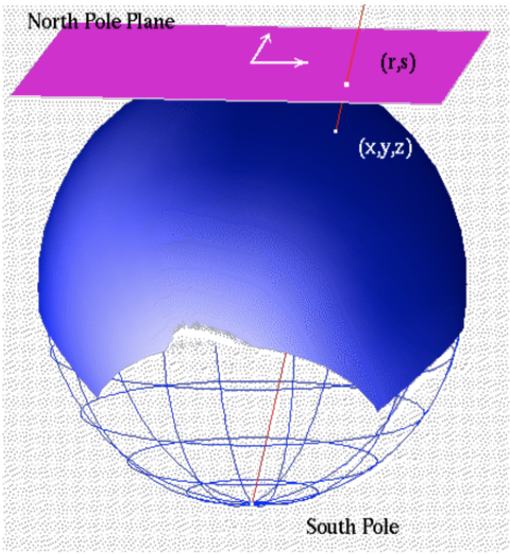
X is called a closed surface if it is compact.

A surface is also sometimes called a 2-*manifold* or a manifold of dimension 2. For any natural number n a topological n-manifold is a Hausdorff topological space X which is locally homeomorphic to $\mathbf{R}^n$.

Remark: (i) The Heine-Borel theorem tells us that a subset of $\mathbf{R}^n$ is compact if and only if it is *closed* (contains all its limit points) and *bounded*. Thus the use of the terminology 'closed surface' for a compact surface is a little perverse: there are plenty of surfaces which are closed subsets of $\mathbf{R}^3$, for example, but which are not 'closed surfaces'.

(ii) Remember that the image of a compact space under a continuous map is always compact, and that a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

Example: The sphere. The most popular way to see that this is a surface according to the definition is stereographic projection:



Here one open set U is the complement of the South Pole and projection identifies it with $\mathbf{R}^2$, the tangent plane at the North Pole. With another open set the complement of the North Pole we see that all points are in a neighbourhood homeomorphic to $\mathbf{R}^2$.

We constructed other surfaces by identification at the boundary of a planar figure. Any subset of the plane has a topology but we need to define one on the space obtained by identifying points. The key to this is to regard identification as an *equivalence relation*. For example, in constructing the torus from the square we define $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ and every other equivalence is an equality. The torus is the set of *equivalence classes* and we give this a topology as follows:

Definition 3 *Let $\sim$ be an equivalence relation on a topological space X. If $x \in X$ let $[x]_\sim = \{y \in X : y \sim x\}$ be the equivalence class of x and let*

$$X/\!\sim = \{[x]_\sim : x \in X\}$$

be the set of equivalence classes. Let $\pi : X \to X/\!\sim$ be the 'quotient' map which sends an element of X to its equivalence class. Then the quotient topology on $X/\!\sim$ is given by

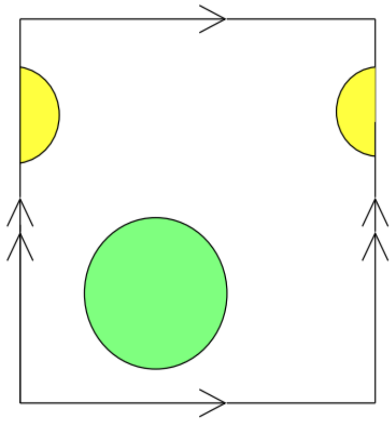
$$\{V \subseteq X/\!\sim : \pi^{-1}(V) \text{ is an open subset of } X\}.$$

In other words a subset V of $X/\sim$ is an open subset of $X/\sim$ (for the quotient topology) if and only if its inverse image

$$\pi^{-1}(V) = \{x \in X : [x]_\sim \in V\}$$

is an open subset of X.

So why does the equivalence relation on the square give a surface? If a point lies inside the square we can take an open disc around it still in the interior of the square. There is no identification here so this neighbourhood is homeomorphic to an open disc in $\mathbf{R}^2$. If the chosen point lies on the boundary, then it is contained in two half-discs D_L, D_R on the left and right:



We need to prove that the quotient topology on these two half-discs is homeomorphic to a full disc. First take the closed half-discs and set $B = D_L \cup D_R$. The map $x \mapsto x + 1$ on D_L and $x \mapsto x$ on D_R is a continuous map from B (with its topology from $\mathbf{R}^2$) to a single disc D. Moreover equivalent points go to the same point so it is a composition

$$B \to B/\sim \to D.$$

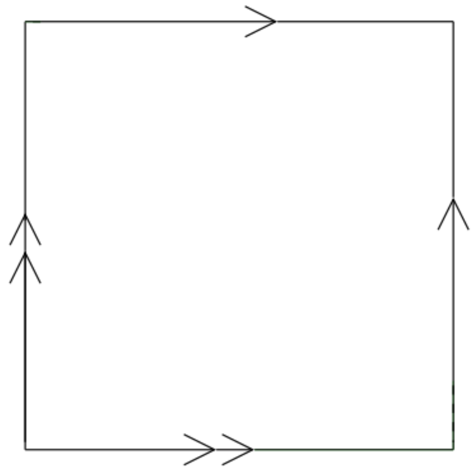
The definition of the quotient topology tells us that $B/\sim \to D$ is continuous. It is also bijective and $B/\sim$, the continuous image of the compact space B, is compact so this is a homeomorphism. Restrict now to the interior and this gives a homeomorphism from a neighbourhood of a point on the boundary of the square to an open disc.

If the point is a corner, we do a similar argument with quadrants.

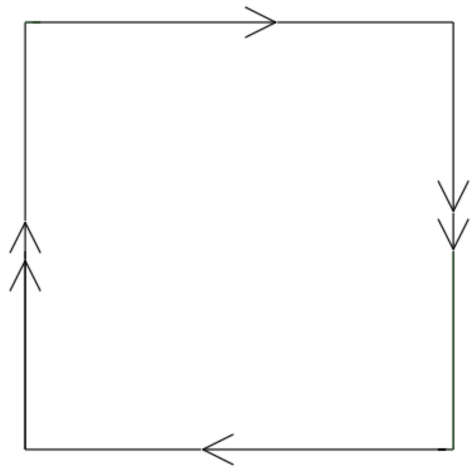
Thus the torus defined by identification is a surface. Moreover it is closed, since it is the quotient of the unit square which is compact.

Here are more examples by identification of a square:

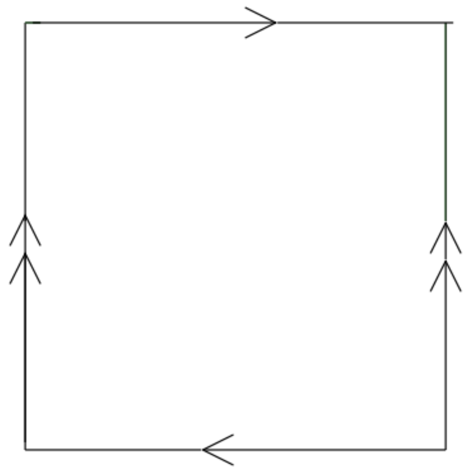
- *The sphere*



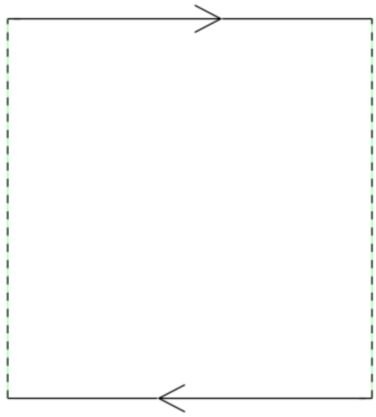
- *Projective space*



- *The Klein bottle*



- *The Möbius band*



The Möbius band is not closed, as the dotted lines suggest. Here is its rigorous definition:

Definition 4 *A Möbius band (or Möbius strip) is a surface which is homeomorphic to*

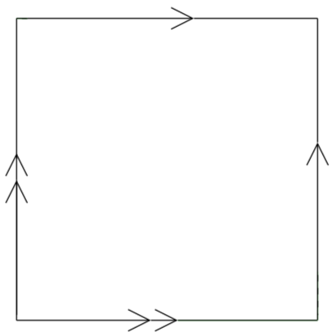
$$(0,1) \times [0,1]/\sim$$

with the quotient topology, where $\sim$ is the equivalence relation given by

$$(x,y) \sim (s,t) \text{ iff } (x = s \text{ and } y = t) \text{ or } (x = 1-s \text{ and } \{y,t\} = \{0,1\}).$$

2.2 Planar models and connected sums

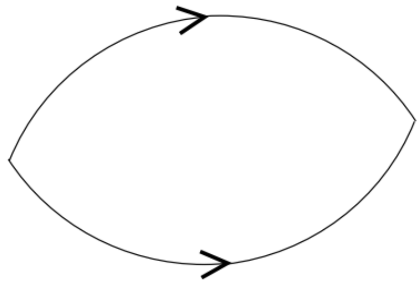
The examples above are obtained by identifying edges of a square but we can use any polygon in the plane with an even number of sides to construct a closed surface so long as we prescribe the way to identify the sides in pairs. Drawing arrows then becomes tiresome so we describe the identification more systematically: going round clockwise we give each side a letter a say, and when we encounter the side to be identified we call it a if the arrow is in the same clockwise direction and a^{-1} if it is the opposite. For example, instead of



we call the top side a and the bottom b and get

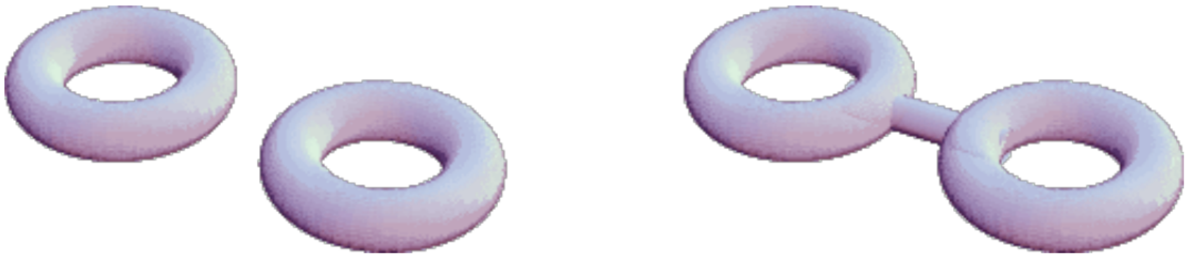
$$aa^{-1}bb^{-1}.$$

This is the sphere. Projective space is then $abab$, the Klein bottle $abab^{-1}$ and the torus $aba^{-1}b^{-1}$. Obviously the cyclic order is not important. There are lots of planar models which define the same surface. The sphere for example can be defined not just from the square but also by aa^{-1}, a 2-sided polygon:



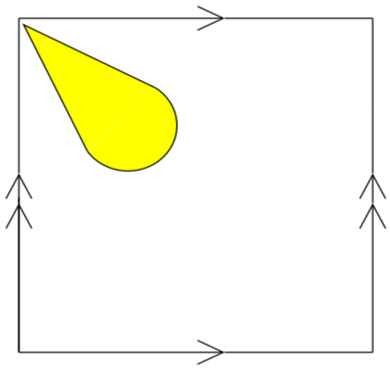
and similarly the projective plane is aa.

Can we get new surfaces by taking more sides? Certainly, but first let's consider another construction of surfaces. If X and Y are two closed surfaces, remove a small open disc from each. Then take a homeomorphism from the boundary of one disc to the boundary of the other. The topological space formed by identifying the two circles is also a surface called the *connected sum* $X\#Y$. We can also think of it as joining the two by a cylinder:

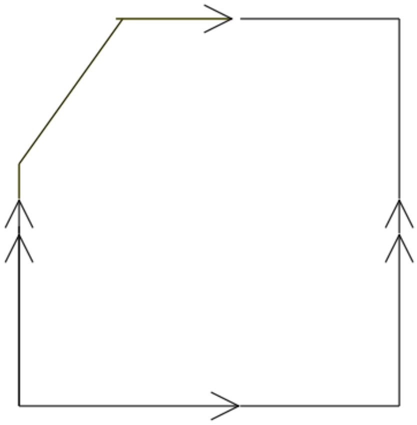


The picture shows that we can get a surface with two holes from the connected sum of two tori. Let's look at this now from the planar point of view.

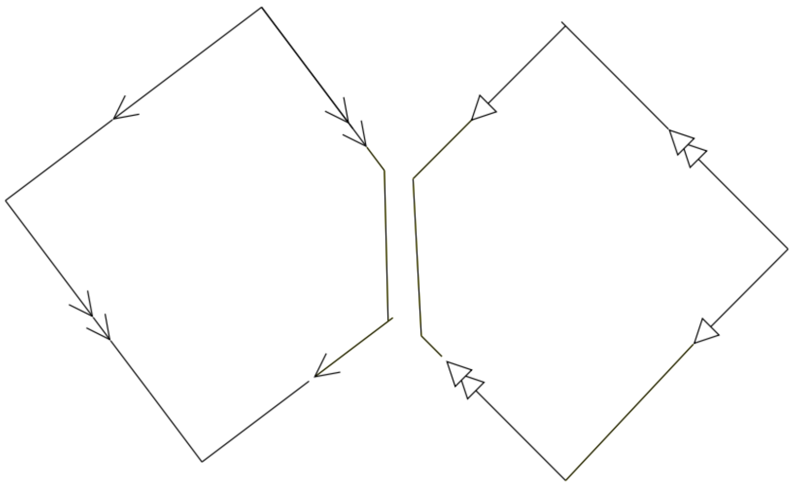
First remove a disc whose boundary passes through a vertex but otherwise misses the sides:



Now open it out:



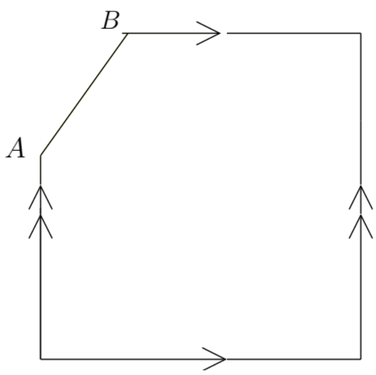
and paste two copies together:



This gives an octagon, and the identification is given by the string of letters:

$$aba^{-1}b^{-1}cdc^{-1}d^{-1}.$$

Let's look more closely at what we have done here. The open disc we removed from the square had no points on the boundary where the equivalence plays a role, so we removed an open disc from the torus. On the other hand "opening it out" means that we also removed the vertex and then replaced it by two points A and B.



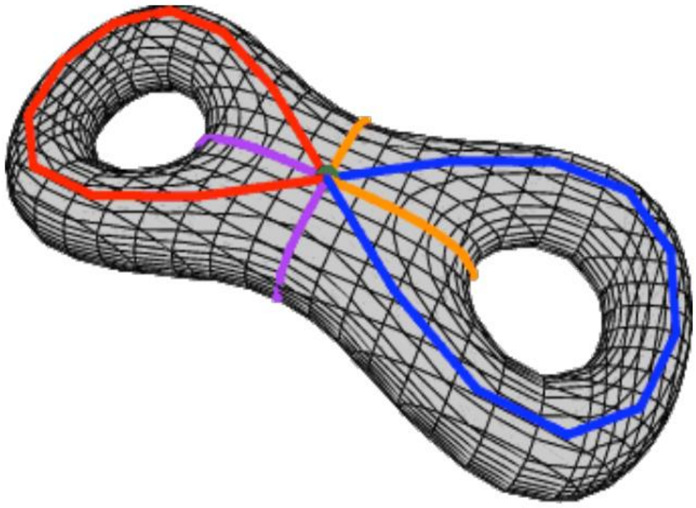
But all four vertices of the square belonged to the same equivalence class, so the removed vertex is still represented on the torus by the equivalence class of the other three. Putting in A, B in the picture above and performing the indicated identifications turns all of the vertices to a single point. So it is a planar model which represents a torus with an open disc removed. The boundary circle of the disc is now the edge AB with end points identified. So the octagon really is a model for the connected sum of the two tori.

It's not hard to see that this is the general pattern: a connected sum can be represented by placing the second string of letters after the first. So in particular

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$$

describes a surface in $\mathbf{R}^3$ with g holes.

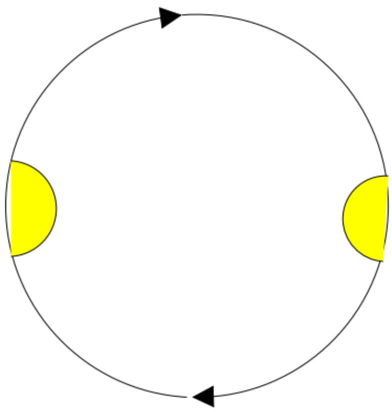
Note that when we defined a torus from a square, all four vertices are equivalent and this persists when we take the connected sum as above. The picture of the surface one should have then is $2g$ closed curves emanating from a single point, and the complement of those curves is homeomorphic to an open disc – the interior of the polygon.



If S is a sphere, then removing a disc just leaves another disc so connected sum with S takes out a disc and replaces it. Thus

$$X \# S = X.$$

Connected sum with the projective plane P is sometimes called *attaching a cross-cap*. In fact, removing a disc from P gives the Möbius band

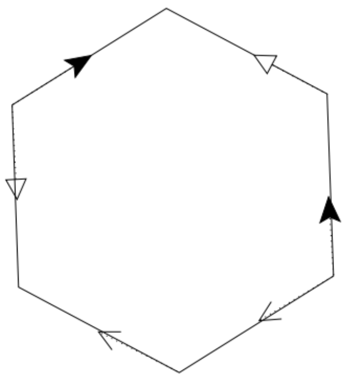


so we are just pasting the boundary circle of the Möbius band to the boundary of the disc. It is easy to see then that the connected sum $P\#P$ is the Klein bottle.

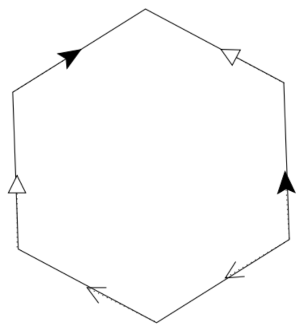
You can't necessarily cancel the connected sum though: it is not true that $X\#A = Y\#A$ implies $X = Y$. Here is an important example:

Proposition 2.1 *The connected sum of a torus T and the projective plane P is homeomorphic to the connected sum of three projective planes.*

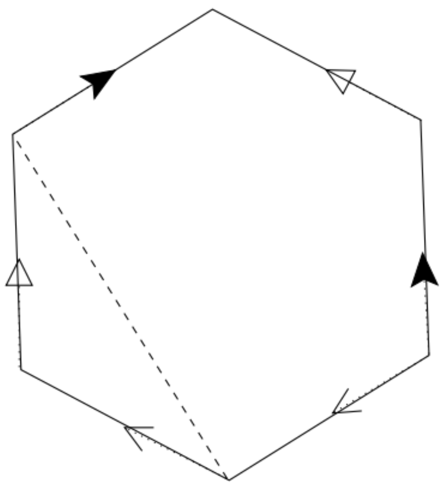
Proof: From the remark above it is sufficent to prove that $P\#T = P\#K$ where K is the Klein bottle. Since P can be described by a 2-gon with relation aa and the Klein bottle is $bcbc^{-1}$, $P\#K$ is defined by a hexagon and the relation $aabcbc^{-1}$.



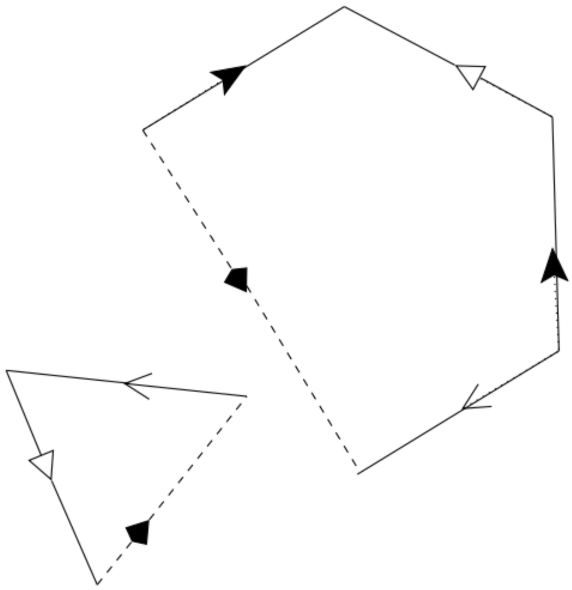
Now $P\#T$ is $aabcb^{-1}c^{-1}$:



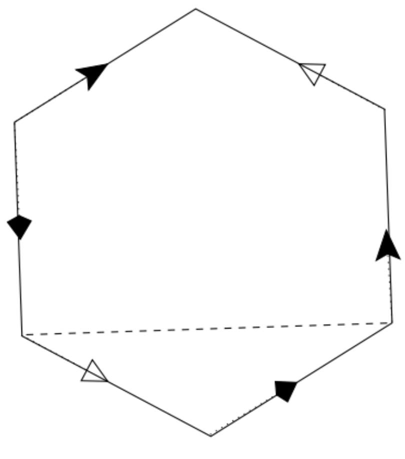
Cut along the dotted line...



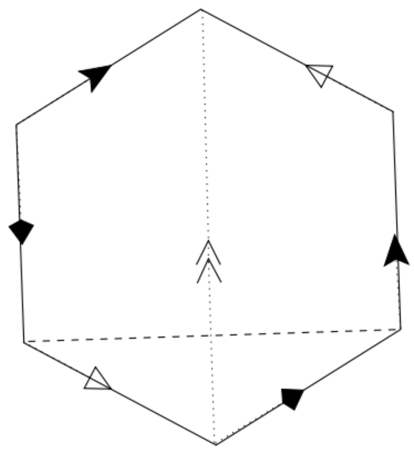
... detach the triangle and turn it over...



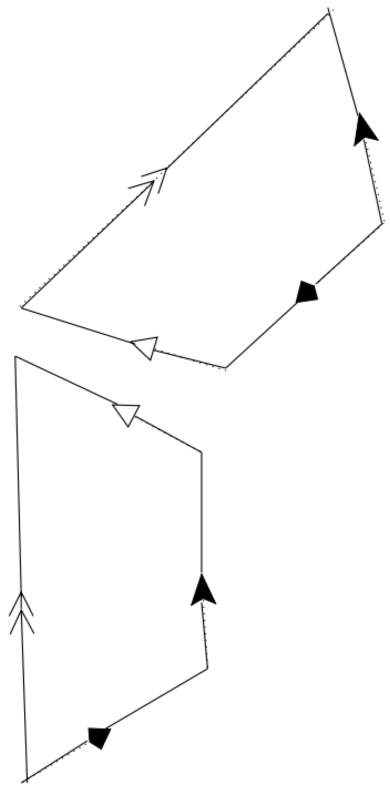
... reattach...



... cut down the middle...



... turn the left hand quadrilateral over and paste together again...



...and this is $aabcbc^{-1}$.

2.3 The classification of surfaces

The planar models allow us to classify surfaces. We shall prove the following

Theorem 2.2 *A closed, connected surface is either homeomorphic to the sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

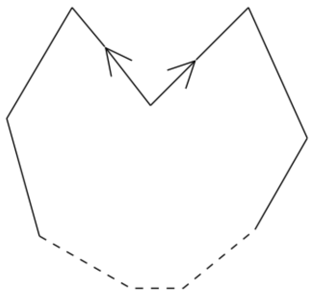
We sketch the proof below (*this is not examinable*) and refer to [2] or [1] for more details. We have to start somewhere, and the topological definition of a surface is quite general, so we need to invoke a theorem beyond the scope of this course: any closed surface X has a *triangulation*: it is homeomorphic to a space formed from the disjoint union of finitely many triangles in $\mathbf{R}^2$ with edges glued together in pairs.

For a Riemann surface (see next section), we can directly find a triangulation so long as we have a meromorphic function, and that is also a significant theorem. If you do a bit more of the differential geometry of surfaces than we do here then the study of geodesics leads to the notion of convex neighbourhoods and you can use geodesic triangles. But both of these use structure beyond the topological definition. Take a look at http://mathoverflow.net/questions/17578/triangulating-surfaces if you want to see an accessible proof.

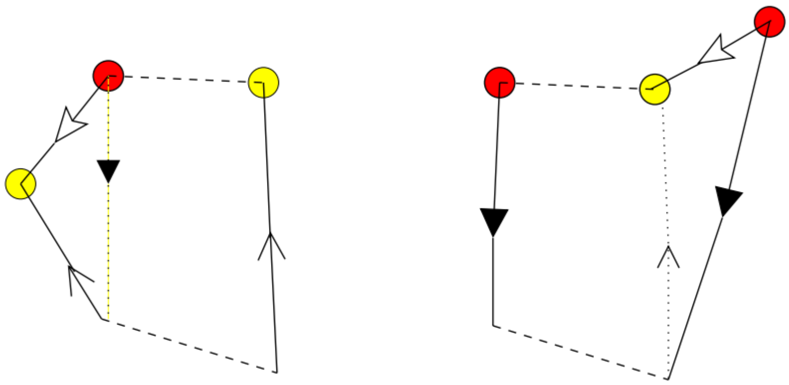
We shall proceed by using a planar model.

Now take one triangle on the surface, and choose a homeomorphism to a planar triangle. Take an adjacent one and the common edge and choose a homeomorphism to another plane triangle and so on... Since the surface is connected the triangles form a polygon and thus X can be obtained from this polygon with edges glued together in pairs. It remains to systematically reduce this, without changing the homeomorphism type, to a standard form.

Step 1: Adjacent edges occurring in the form aa^{-1} or $a^{-1}a$ can be eliminated.

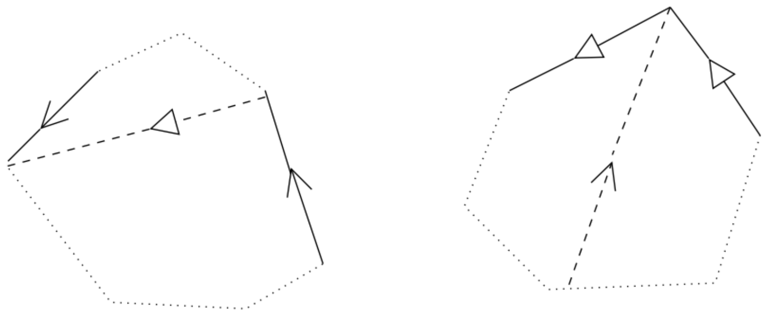


Step 2: We can assume that all vertices must be identified with each other. To see this, suppose Step 1 has been done, and we have two adjacent vertices in different equivalence classes: red and yellow. Because of Step 1 the other side going through the yellow vertex is paired with a side elsewhere on the polygon. Cut off the triangle and glue it onto that side:

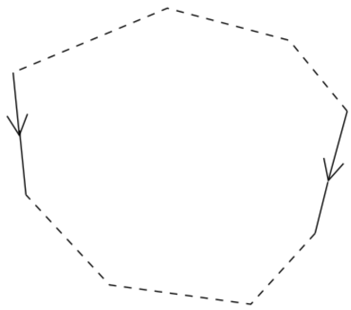


The result is the same number of sides but one less yellow and one more red vertex. Eventually, applying Step 1 again, we get to a single equivalence class.

Step 3: We can assume that any pair of the form a and a are adjacent, by cutting and pasting:

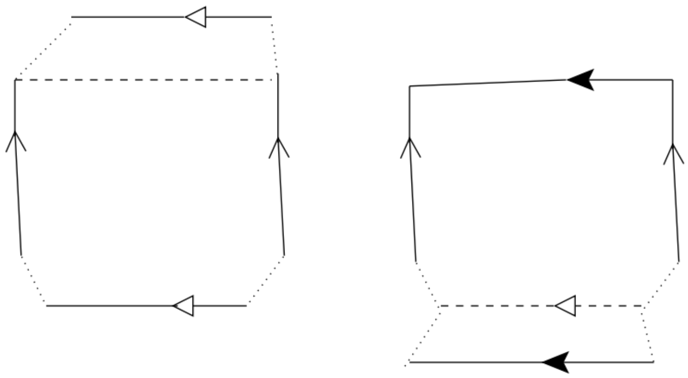


We now have a single equivalence class of vertices and all the pairs a, a are adjacent. What about a pair a, a^{-1}? If they are adjacent, Step 1 gets rid of them, if not we have this:

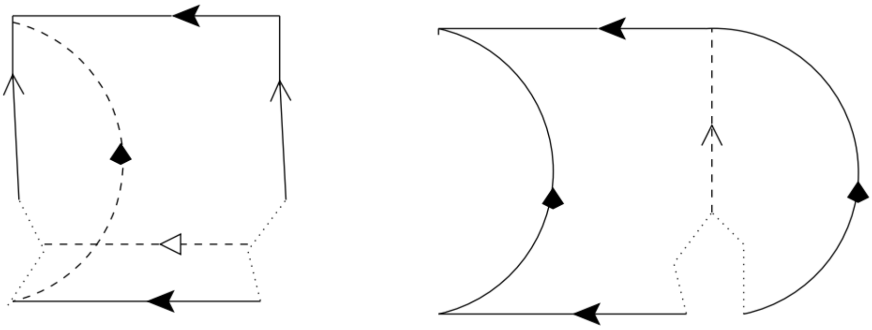


If all the sides on the top part have their partners in the top part, then their vertices will never be equivalent to a vertex in the bottom part. But Step 2 gave us one equivalence class, so there is a b in the top half paired with something in the bottom. It can't be b because Step 3 put them adjacent, so it must be b^{-1}.

Step 4: We can reduce this to something of the form $cdc^{-1}d^{-1}$ like this. First cut off the top and paste it to the bottom.



Now cut away from the left and paste it to the right.



Finally our surface is described by a string of terms of the form aa or $bcb^{-1}c^{-1}$: a connected sum of projective planes and tori. However, if there is at least one projective plane we can use Proposition 2.1 which says that $P\#T = P\#P\#P$ to get rid of the tori.

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