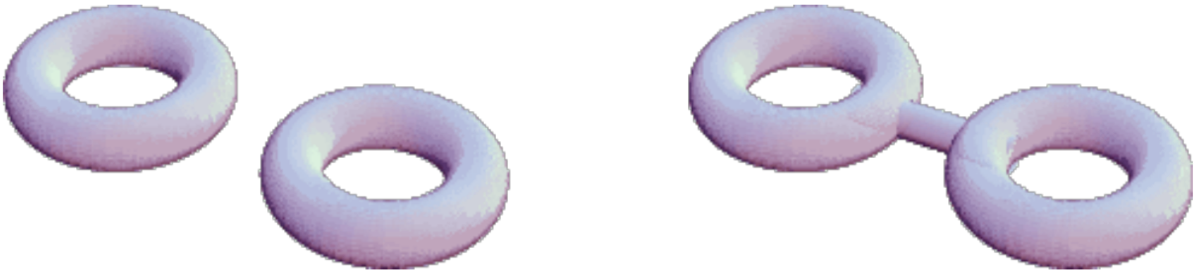
Orientability

Given a surface, we need to be able to decide what connected sum it is in the Classification Theorem without cutting it into pieces. Fortunately there are two concepts, which are invariant under homeomorphism, which do this. The first concerns orientation:

Definition 5 *A surface X is orientable if it contains no open subset homeomorphic to a Möbius band.*

From the definition it is clear that if X is orientable, any surface homeomorphic to X is too.

We saw that taking the connected sum with the projective plane means attaching a Möbius band, so the surfaces which are connected sums of P are non-orientable. We need to show that connected sums of tori are orientable. For this, we observe that the connected sum operation works for tori in $\mathbf{R}^3$ embedded in the standard way:



so a connected sum of tori can also be embedded in $\mathbf{R}^3$. The sketch proof below assumes our surfaces are differentiable – we shall deal with these in more detail later.

Suppose for a contradiction that X is a non-orientable compact smooth surface in $\mathbf{R}^3$. Then X has an open subset which is homeomorphic to a Möbius band, which means that we can find a loop (i.e. a closed path) in X such that the normal to X, when transported around the loop in a continuous fashion, comes back with the opposite direction. By considering a point on the normal a small distance from X, moving it around the loop and then connecting along the normal from one side of X to the other, we can construct a closed path $\gamma : [0, 1] \to \mathbf{R}^3$ in $\mathbf{R}^3$ which meets X at exactly one point and is *transversal* to X at this point (i.e. the tangent to γ at x is not tangent to X). It is a general fact about the topology of $\mathbf{R}^3$ that any closed differentiable path $\gamma : [0, 1] \to \mathbf{R}^3$ can be 'filled in' with a disc; more precisely there is a differentiable map $f : D \to \mathbf{R}^3$, where $D = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 \leq 1\}$, such that

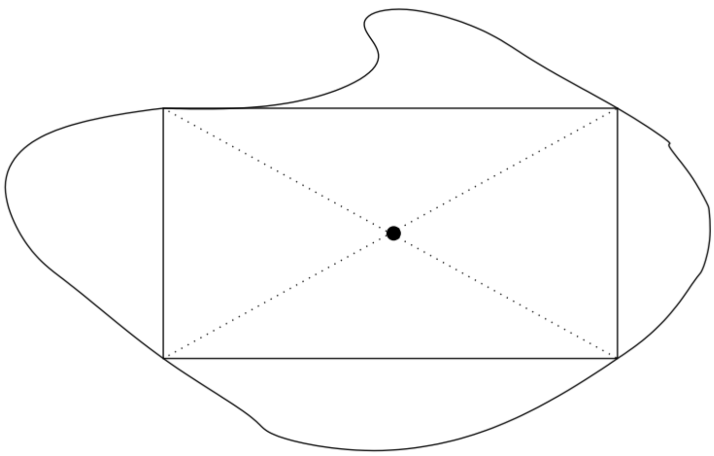
$$\gamma(t) = f(\cos 2\pi t, \sin 2\pi t)$$

for all $t \in [0, 1]$. Now we can perturb f a little bit, without changing γ or the values of f on the boundary of D, to make f transversal to X (i.e. the image of f is not tangent to X at any point of intersection with X). But once f is transversal to X it can be shown that the inverse image $f^{-1}(X)$ of X in D is very well behaved: it consists of a disjoint union of simple closed paths in the interior of D, together with paths meeting the boundary of D in exactly their endpoints (which are two distinct points on the boundary of D). Thus $f^{-1}(X)$ contains an even number of points on

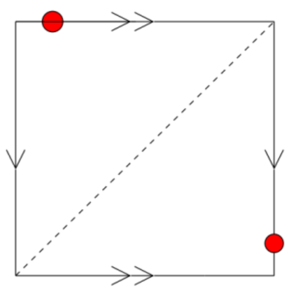
the boundary of D, which contradicts our construction in which $f^{-1}(X)$ has exactly one point on the boundary of D. The surface must therefore be orientable.

This argument shows why the projective plane in particular can't be embedded in $\mathbf{R}^3$. Here is an amusing corollary:

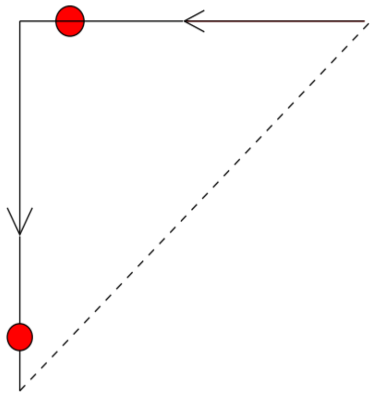
Proposition 2.3 *Any simple closed curve in the plane contains an inscribed rectangle.*



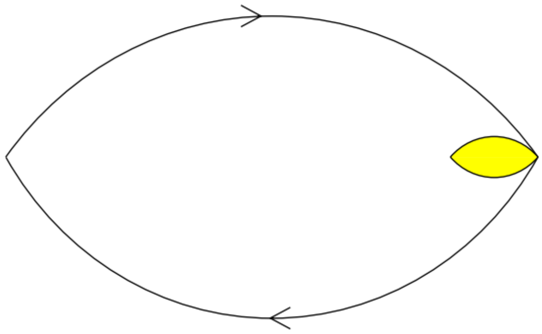
Proof: The closed curve C is homeomorphic to the circle. Consider the set of pairs of points (x, y) in C. This is the product of two circles: a *torus*. We now want to consider the set X of *unordered* pairs, so consider the planar model of the torus. We identify (x, y) with (y, x), which is reflection about the diagonal. The top side then gets identified with the right hand side, and under the torus identification with the left hand side.



The set of unordered points is therefore obtained by identification on the top triangle:



and this is the projective plane with a disc removed (the Möbius band):



Now define a map $f : X \to \mathbf{R}^3$ as follows:

$$(x, y) \mapsto (\frac{1}{2}(x + y), |x - y|) \in \mathbf{R}^2 \times \mathbf{R}$$

The first term is the midpoint of the line xy and the last is the distance between x and y. Both are clearly independent of the order and so the map is well-defined. When $x = y$, which is the boundary circle of the Möbius band, the map is

$$x \mapsto (x, 0)$$

which is the curve C in the plane $x_3 = 0$. Since the curve bounds a disc we can extend f to the surface obtained by pasting the disc to X and extending f to be the inclusion of the disc into the plane $x_3 = 0$. This is a continuous map (it can be perturbed to be differentiable if necessary) of the projective plane P to $\mathbf{R}^3$. Since P is unorientable it can't be an embedding so we have at least two pairs $(x_1, y_1), (x_2, y_2)$ with the same centre and the same separation. These are the vertices of the required rectangle. $\square$

The Euler characteristic

It is a familar fact (already known to Descartes in 1639) that if you divide up the surface of a sphere into polygons and count the number of vertices, edges and faces then

$$V - E + F = 2.$$

This number is the Euler characteristic, and we shall define it for any surface. First we have to define our terms:

Definition 6 *A subdivision of a compact surface X is a partition of X into*

i) vertices *(these are finitely many points of X),*

ii) edges *(finitely many disjoint subsets of X each homeomorphic to the open interval $(0, 1)$), and*

iii) faces *(finitely many disjoint open subsets of X each homeomorphic to the open disc $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$ in $\mathbf{R}^2$,*

such that

a) the faces are the connected components of $X \setminus \{$vertices and edges$\}$,

b) no edge contains a vertex, and

c) each edge 'begins and ends in a vertex' (either the same vertex or different vertices), or more precisely, if e is an edge then there are vertices v_0 and v_1 (not necessarily distinct) and a continuous map

$$f : [0, 1] \to e \cup \{v_0, v_1\}$$

which restricts to a homeomorphism from $(0, 1)$ to e and satisfies $f(0) = v_0$ and $f(1) = v_1$.

Definition 7 *The Euler characteristic (or Euler number) of a compact surface X with a subdivision is*

$$\chi(X) = V - E + F$$

where V is the number of vertices, E is the number of edges and F is the number of faces in the subdivision.

The fact that a closed surface has a subdivision follows from the existence of a triangulation. The most important fact is

Theorem 2.4 *The Euler characteristic of a compact surface is independent of the subdivision*

which we shall sketch a proof of later. Note that we can define a subdivision for more general topological spaces than closed surfaces, for example a triangle has one face, 3 vertices and 3 edges and hence Euler characteristic equal to 1.

A planar model provides a subdivision of a surface. We have one face – the interior of the polygon – and if there are $2n$ sides to the polygon, these get identified in pairs so there are n edges. For the vertices we have to count the number of equivalence classes, but in the normal form of the classification theorem, we created a single equivalence class. In that case, the Euler characteristic is

$$1 - n + 1 = 2 - n.$$

The connected sum of g tori had $4g$ sides in the standard model $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ so in that case $\chi(X) = 2 - 2g$. The connected sum of g projective planes has $2g$ sides so we have $\chi(X) = 2 - g$. We then obtain:

Theorem 2.5 *A closed surface is determined up to homeomorphism by its orientability and its Euler characteristic.*

This is a very strong result: nothing like this happens in higher dimensions.

To calculate the Euler characteristic of a given surface we don't necessarily have to go to the classification. Suppose a surface is made up of the union of two spaces X and Y, such that the intersection $X \cap Y$ has a subdivision which is a subset of the subdivisions for X and for Y. Then since V, E and F are just counting the number of elements in a set, we have immediately that

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y).$$

We can deal with a connected sum this way. Take a closed surface X and remove a disc D to get a space X^o. The disc has Euler characteristic 1 (a polygon has one face,

n vertices and n sides) and the boundary circle has Euler characteristic 0 (no face). So applying the formula,

$$\chi(X) = \chi(X^o \cup D) = \chi(X^o) + \chi(D) - \chi(X^o \cap D) = \chi(X^o) + 1.$$

To get the connected sum we paste X^o to Y^o along the boundary circle so

$$\chi(X\#Y) = \chi(X^o)+\chi(Y^o)-\chi(X^o \cap Y^o) = \chi(X)-1+\chi(Y)-1-0 = \chi(X)+\chi(Y)-2.$$

In particular, $\chi(X\#T) = \chi(X) - 2$ so this again gives the value $2 - 2g$ for the connected sum of g tori.

To make all this work we finally need:

Theorem 2.6 *The Euler characteristic $\chi(X)$ of a compact surface X is a topological invariant.*

We give a sketch proof (which is not examinable).

Proof:

The idea is to give a different definition of $\chi(X)$ which makes it clear that it is a topological invariant, and then prove that the Euler characteristic of any subdivision of X is equal to $\chi(X)$ defined in this new way.

For each continuous path $f : [0,1] \to X$ define its boundary ∂f to be the formal linear combination of points $f(0) + f(1)$. If g is another map and $g(0) = f(1)$ then, with coefficients in $\mathbf{Z}/2$, we have

$$\partial f + \partial g = f(0) + 2f(1) + g(1) = f(0) + g(1)$$

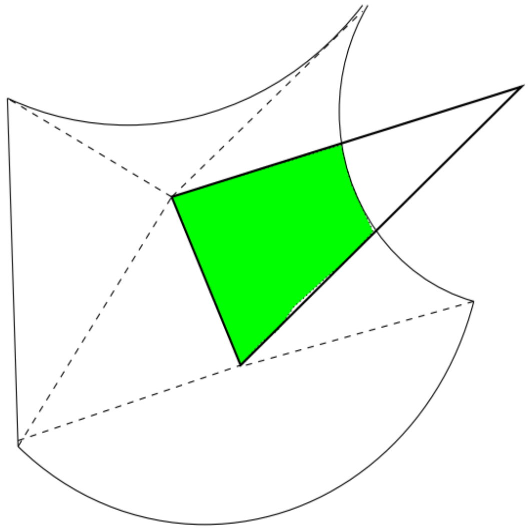
which is the boundary of the path obtained by sticking these two together. Let C_0 be the vector space of finite linear combinations of points with coefficients in $\mathbf{Z}/2$ and C_1 the linear combinations of paths, then $\partial : C_1 \to C_0$ is a linear map. If X is connected then any two points can be joined by a path, so that $x \in C_0$ is in the image of ∂ if and only if it has an even number of terms.

Now look at continuous maps of a triangle $ABC = \Delta$ to X and the space C_2 of all linear combinations of these. The boundary of $F : \Delta \to X$ is the sum of the three paths which are the restrictions of F to the sides of the triangle. Then

$$\partial\partial F = (F(A) + F(B)) + (F(B) + F(C)) + (F(C) + F(A)) = 0$$

so that the image of $\partial : C_2 \to C_1$ is contained in the kernel of $\partial : C_1 \to C_0$. We define $H_1(X)$ to be the quotient space. This is clearly a topological invariant because we only used the notion of continuous functions to define it.

If we take X to be a surface with a subdivision, one can show that because each face is homeomorphic to a disc, any element in the kernel of $\partial : C_1 \to C_0$ can be replaced by a linear combination of edges of the subdivision upon adding something in ∂C_2 :



Now we let $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$ be vector spaces over $\mathbf{Z}/2$ with bases given by the sets of vertices, edges and faces of the subdivision, then define boundary maps in the same way

$$\partial : \mathcal{E} \to \mathcal{V} \text{ and } \partial : \mathcal{F} \to \mathcal{E}.$$

Then

$$H_1(X) \cong \frac{\ker(\partial : \mathcal{E} \to \mathcal{V})}{\operatorname{im}(\partial : \mathcal{F} \to \mathcal{E})}.$$

By the rank-nullity formula we get

$$\dim H_1(X) = \dim \mathcal{E} - \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V}) - \dim \mathcal{F} + \dim \ker(\partial : \mathcal{F} \to \mathcal{E}).$$

Because X is connected the image of $\partial : \mathcal{E} \to \mathcal{V}$ consists of sums of an even number of vertices so that

$$\dim \mathcal{V} = 1 + \operatorname{rk}(\partial : \mathcal{E} \to \mathcal{V}).$$

Also $\ker(\partial : \mathcal{F} \to \mathcal{E})$ is clearly spanned by the sum of the faces, hence

$$\dim \ker(\partial : \mathcal{F} \to \mathcal{E}) = 1$$

so

$$\dim H_1(X) = 2 - V + E - F.$$

This shows that $V - E + F$ is a topological invariant. □

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