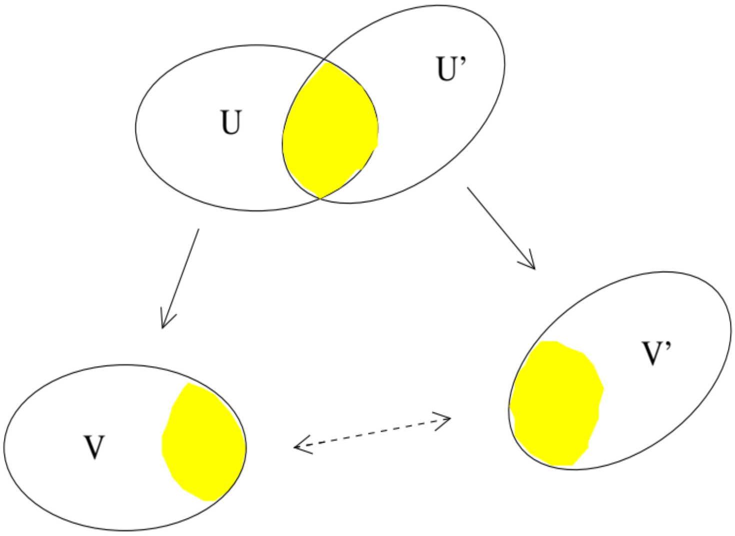
Riemann surfaces

Definitions and examples

From the definition of a surface, each point has a neighbourhood U and a homeomorphism φ_U from U to an open set V in $\mathbf{R}^2$. If two such neighbourhoods U, U' intersect, then

$$\varphi_{U'}\varphi_U^{-1} : \varphi_U(U \cap U') \to \varphi_{U'}(U \cap U')$$

is a homeomorphism from one open set of $\mathbf{R}^2$ to another.



If we identify $\mathbf{R}^2$ with the complex numbers $\mathbf{C}$ then we can define:

Definition 8 *A Riemann surface is a surface with a class of homeomorphisms φ_U such that each map $\varphi_{U'}\varphi_U^{-1}$ is a* holomorphic *(or* analytic*) homeomorphism.*

We call each function φ_U a holomorphic coordinate.

In your course on complex analysis you used holomorphic functions in two ways: one involved adding, multiplying, differentiating and taking contour integrals; the other concerned conformal mappings, taking one domain to another, generally in order to simplify a contour integral. It is this second viewpoint which we use in this definition.

Examples:

1. Let X be the extended complex plane $X = \mathbf{C} \cup \{\infty\}$. Let $U = \mathbf{C}$ with $\varphi_U(z) = z \in \mathbf{C}$. Now take

$$U' = \mathbf{C}\backslash\{0\} \cup \{\infty\}$$

and define $z' = \varphi_{U'}(z) = z^{-1} \in \mathbf{C}$ if $z \neq \infty$ and $\varphi_{U'}(\infty) = 0$. Then

$$\varphi_U(U \cap U') = \mathbf{C}\backslash\{0\}$$

and

$$\varphi_U \varphi_{U'}^{-1}(z) = z^{-1}$$

which is holomorphic.

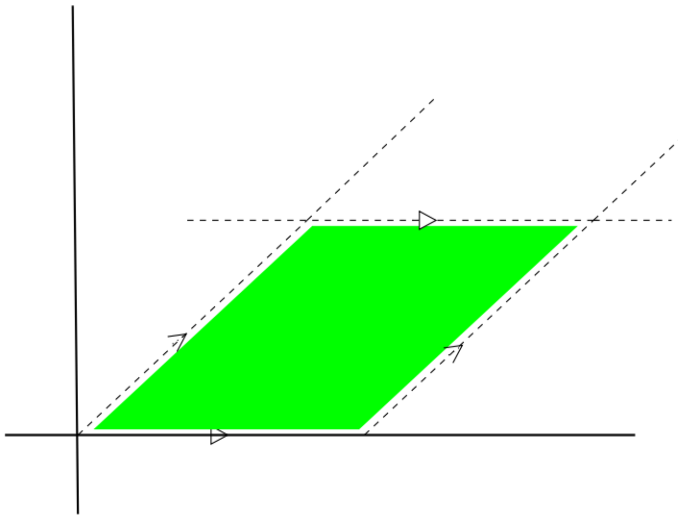
In the right coordinates this is the sphere, with ∞ the North Pole and the coordinate maps given by stereographic projection. For this reason it is sometimes called the *Riemann sphere.*

2. Let $\omega_1, \omega_2 \in \mathbf{C}$ be two complex numbers which are linearly independent over the reals, and define an equivalence relation on $\mathbf{C}$ by $z_1 \sim z_2$ if there are integers m, n such that $z_1 - z_2 = m\omega_1 + n\omega_2$. Let X be the set of equivalence classes (with the quotient topology). A small enough disc V around $z \in \mathbf{C}$ has at most one representative in each equivalence class, so this gives a local homeomorphism to its projection U in X. If U and U' intersect, then the two coordinates are related by a map

$$z \mapsto z + m\omega_1 + n\omega_2$$

which is holomorphic.

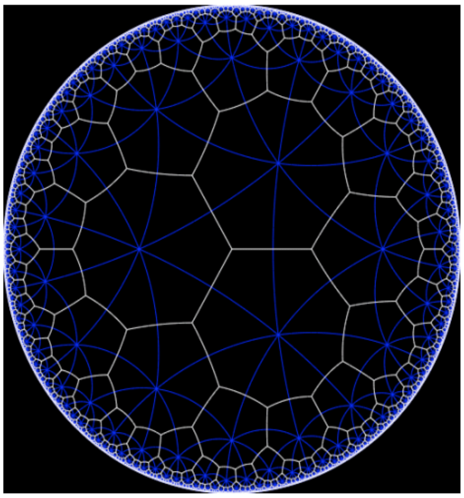
This surface is topologically described by noting that every z is equivalent to one inside the closed parallelogram whose vertices are $0, \omega_1, \omega_2, \omega_1 + \omega_2$, but that points on the boundary are identified:



We thus get a torus this way. Another way of describing the points of the torus is as *orbits* of the action of the group $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{C}$ by $(m, n) \cdot z = z + m\omega_1 + n\omega_2$.

3. The parallelograms in Example 2 fit together to tile the plane. There are groups of holomorphic maps of the unit disc into itself for which the interior of a polygon

plays the same role as the interior of the parallelogram in the plane, and we get a surface X by taking the orbits of the group action. Now we get a tiling of the disc:



In this example the polygon has eight sides and the surface is homeomorphic by the classification theorem to the connected sum of two tori.

4. A *complex algebraic curve* X in $\mathbf{C}^2$ is given by

$$X = \{(z, w) \in \mathbf{C}^2 : f(z, w) = 0\}$$

where f is a polynomial in two variables with complex coefficients. If $(\partial f/\partial z)(z, w) \neq 0$ or $(\partial f/\partial w)(z, w) \neq 0$ for every $(z, w) \in X$, then using the implicit function theorem (see Appendix A) X can be shown to be a Riemann surface with local homeomorphisms given by

$$(z, w) \mapsto w \text{ where } (\partial f/\partial z)(z, w) \neq 0$$

and

$$(z, w) \mapsto z \text{ where } (\partial f/\partial w)(z, w) \neq 0.$$

Definition 9 *A holomorphic map between Riemann surfaces X and Y is a continuous map $f : X \to Y$ such that for each holomorphic coordinate φ_U on U containing x on X and ψ_W defined in a neighbourhood of $f(x)$ on Y, the composition*

$$\psi_W \circ f \circ \varphi_U^{-1}$$

is holomorphic.

In particular if we take $Y = \mathbf{C}$, we can define holomorphic functions on X, and then we can use the ring structure of $\mathbf{C}$ to add and multiply these functions.

Before proceeding, recall some basic facts about holomorphic functions (see [3]):

- A holomorphic function has a convergent power series expansion in a neighbourhood of each point at which it is defined:

$$f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots$$

- If f vanishes at c then

$$f(z) = (z-c)^m(c_0 + c_1(z-c) + \dots)$$

  where $c_0 \neq 0$. In particular zeros are isolated.

- If f is non-constant it maps open sets to open sets.

- $|f|$ cannot attain a maximum at an interior point of a disc ("maximum modulus principle").

- $f : \mathbf{C} \mapsto \mathbf{C}$ preserves angles between differentiable curves, both in magnitude and sense.

This last property shows:

Proposition 3.1 *A Riemann surface is orientable.*

Proof: Assume X contains a Möbius band, and take a smooth curve down the centre: $\gamma : [0,1] \to X$. In each small coordinate neighbourhood of a point on the curve $\varphi_U \gamma$ is a curve in a disc in $\mathbf{C}$, and rotating the tangent vector γ' by $90°$ or $-90°$ defines an upper and lower half:

Identification on an overlapping neighbourhood is by a map which preserves angles, and in particular the sense – anticlockwise or clockwise – so the two upper halves agree on the overlap, and as we pass around the closed curve the strip is separated into two halves. But removing the central curve of a Möbius strip leaves it connected:

which gives a contradiction. □

From the classification of surfaces we see that a closed, connected Riemann surface is homeomorphic to a connected sum of tori.

The local form of a holomorphic map between Riemann surfaces

Theorem (Local form of a holomorphic map). *For any holomorphic map $f : S \to R$ between Riemann surfaces, with $f(s) = r$, we can choose local complex coordinates around $s \in S$, $r \in R$, so that in local coordinates f is the map*

$$f : D \to D,\ f(z) = z^n.$$

Proof. We can assume (by translating) that the local coordinates are chosen so that s, r correspond to $0 \in \mathbb{C}$, so in local coordinates $f(0) = 0$. The Taylor series for f is

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots = a_n z^n (1 + \text{higher order terms})$$

[1]We need to check that in a small neighbourhood of a point (z, w) for large $z \neq \infty$, the transition map from the local coordinate z to the local coordinate Y is biholomorphic. But near (X, Y) with $X \neq 0$ we can use either X or Y as local coordinate since we can express Y in terms of a holomorphic branch of the square root of a polynomial in X (this is the same argument that proves that for the Riemann surface $w^2 = z$ you may declare that w is a holomorphic coordinate near $(w, z) = (0, 0)$, and z is a holomorphic coordinate elsewhere). Thus it suffices to find a holomorphic transition from z to X, away from $X = 0$. But that we know: $z \mapsto X = 1/z$ is the required local biholomorphism in z.

[2]To understand the cut: for $w^2 = (z-1)(z-2)$, why do we cut the segment $(1, 2)$? The local model near $z = 1$ and near $z = 2$ is that of the square root, and for the square root we typically choose the cut along the negative real axis. In our case, we make cuts $(-\infty, 1)$ and $(-\infty, 2)$ and we pick branches of $\sqrt{z-1}$ and of $\sqrt{z-2}$. For one copy of $\mathbb{C}$, for $z = 1 + ae^{i\theta} = 2 + be^{i\psi}$ let's declare $\sqrt{z-1} = a^{1/2} e^{i\theta/2}$ and $\sqrt{z-2} = b^{1/2} e^{i\psi/2}$ for $\theta, \psi \in (-\pi, \pi)$. We would think that this is only acceptable if there is a cut along $(-\infty, 2) \subset \mathbb{R} \subset \mathbb{C}$, but in fact for real $z < 1$ the two discontinuities cancel out: $e^{i\pi/2} e^{i\pi/2} = e^{i\pi} = e^{-i\pi} = e^{i(-\pi/2)} e^{i(-\pi/2)}$.

[3]If you are curious about why these clever coordinates work at infinity, unlike the projectivization which typically gives rise to a singularity at infinity, then have a look at http://en.wikipedia.org/wiki/Hyperelliptic_curve

[4]Explicitly and pedantically: there are local parametrizations $F : V \to S$, $0 \in V \subset \mathbb{R}^2$, $G : W \to R$, $0 \in W \subset \mathbb{R}^2$, $F(0) = s$, $G(0) = r$ and $f^{\text{local}}(z) = G^{-1} \circ f \circ F(z) = z^n$. We abusively just say "locally $f = \ldots$".

where $a_n \neq 0$ is the first non-zero coefficient. This $n \geq 1$ is called the order of the vanishing $f(0) = 0$. A holomorphic n-th root of f is then defined[1] near 0, with $f(z)^{1/n} = a_n^{1/n} z +$ higher terms. The derivative of this n-th root at $z = 0$ is $a_n^{1/n} \neq 0$. By the inverse function theorem, there is a local holomorphic inverse $G : \mathbb{C} \to \mathbb{C}$ defined near 0, so $G(0) = 0$ and

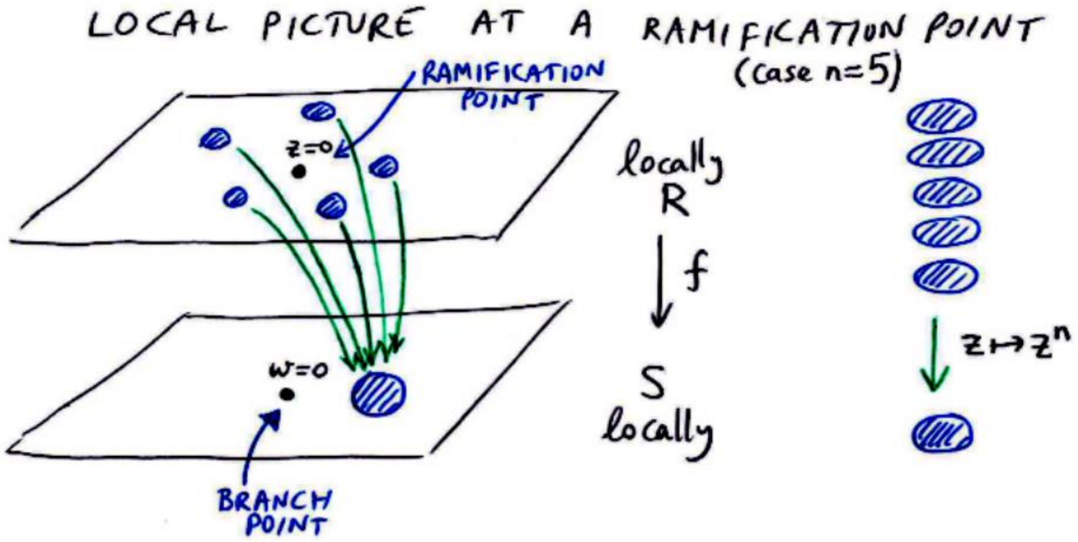
$$f(G(z))^{1/n} = z.$$

Now we can change coordinates on the domain using the local biholomorphism G, explicitly: if $F : \mathbb{C} \to S$ (defined near $0 \in \mathbb{C}$) was the original local parametrization near $s \in S$, then the new one is $F \circ G : \mathbb{C} \to \mathbb{C} \to S$ (defined near $0 \in \mathbb{C}$). The new local expression for f becomes

$$z \mapsto f(G(z)) = z^n. \quad \square$$

Branch points and ramification points

Recall the local form Theorem [ ] $f : R \to S$ has the local form $z \mapsto z^n$ near $p \in R$, where p corresponds to $z = 0$ locally.



[1]For $w \in \mathbb{C}$ with $|w| < 1$, $(1+w)^a$ can be expanded in a binomial series

$$(1+w)^a = 1 + aw + \frac{a(a-1)}{2!}w^2 + \frac{a(a-1)(a-2)}{3!}w^3 + \cdots$$

for any $a \in \mathbb{C}$, and the series converges absolutely.

We call

$$v_f(p) = n$$

the **valency** of f at p. Geometrically, it tells you how many solutions there are to the equation

$$f(z) = w$$

for small $w \neq 0$. This shows n does not depend on the choice of local coordinates near p, $f(p)$. The point $w = 0$ is bad because there are missing solutions: only $z = 0$ is a solution for small z. We call $z = 0$ a ramification point and $w = 0$ a branch point. Intuitively, a ramification point in R is where the local number of solutions of $f(z) = w$ has suddenly dropped. The value of w, when solutions are missing, is a branch point.

Definition (Ramification point and branch point). *For a point p where $v_f(p) \neq 1$:*

(1) *$p \in R$ is called **ramification point***
(2) *the image $f(p) \in S$ is called **branch point***
(3) *$v_f(p)$ is called **ramification index***

Equivalently:
◇ *$r \in R$ is a **ramification point** $\iff$ the derivative $f'(r) = 0$ in local coordinates,*
◇ *$s \in S$ is a **branch point** $\iff$ the preimage $f^{-1}(s) \subset R$ contains a ramification point,*
◇ *the **ramification index** $= 1 +$ number of derivatives of f vanishing at r in local coordinates.*

Lemma *For compact R, $v_f(p) = 1$ for all except finitely many $p \in R$.*

Proof. Locally $f(z) = z^n$, so $f'(z) = nz^{n-1} \neq 0$ for $z \neq 0$ near $z = 0$. So the subset in R of points where $v_f(p) > 1$ is discrete. So it is finite when R is compact.

The degree of a holomorphic map of compact Riemann surfaces

Definition (Degree)**.** *The* ***degree*** *of a non-constant holomorphic map $f : R \to S$ between compact connected Riemann surfaces is*

$$\deg(f) = \sum_{r \in f^{-1}(s)} v_f(r)$$

where we fix a point $s \in S$.

Theorem **.** $\deg(f)$ *does not depend on the choice of point $s \in S$.*

Proof. Since R is compact, $f^{-1}(s) \subset R$ is finite (f is not[1] constantly r). Pick small disjoint discs $D_p \subset R$, one around each point $p \in f^{-1}(s)$, so that on each disc f has the local form $z \mapsto z^{v_f(p)}$. By shrinking the radii of the discs D_p, we can assume all D_p map surjectively onto the same open neighbourhood V of s. By construction, $f^{-1}(V)$ contains all the discs D_p, but may contain also other points of R. However, by further shrinking the radii of the D_p we can ensure that $f^{-1}(V) = \cup D_p$ contains nothing else.[2]

By the local model, it follows that $d = \sum_{p \in f^{-1}(s)} v_f(p)$ is the number of solutions to the equation $f(q) = w$ for $w \neq s \in V$ (in each disc D_p we find $v_f(p)$ solutions). Thus

$$|f^{-1}(w)| = \sum_{q \in f^{-1}(w)} v_f(q) = d,$$

[1]If $f^{-1}(s)$ had infinitely many points, then it would have a limit point r. At this limit point r you could not have a local form of type $z \mapsto z^N$, as s has infinitely many preimages near r (in particular more than N).

[2]otherwise, by contradiction, there would be a sequence of points in R bounded away from $\cup\{p\} = f^{-1}(s)$ for which the f-values converge to s. By compactness of R a subsequence would converge to a point p' with $f(p') = s$ which we had not included in $\cup\{p\} = f^{-1}(s)$, contradiction.

independently of the choice of $w \neq 0 \in V$. Since S is connected, the locally constant function $w \mapsto \sum_{q \in f^{-1}(w)} v_f(q)$ on S must be constant. □

Corollary *For all points $s \in S$ except branch points, there are precisely* $\deg(f) = |f^{-1}(s)|$ *points in R mapping to s.*

Example. For a complex polynomial $f(z)$ of degree d, we have $d = \deg(f)$. So there are precisely d solutions to $f(z) = 0$ unless 0 is a branch point of f. When 0 is a branch point, there are repeated roots, but if we count the roots with the correct multiplicity $v_f(p)$ then there are still d solutions.

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