Meromorphic functions

Recall that on a closed (i.e. compact) surface X, any continuous real function achieves its maximum at some point x. Let X be a Riemann surface and f a holomorphic function, then $|f|$ is continuous, so assume it has its maximum at x. Since $f\varphi_U^{-1}$ is a holomorphic function on an open set in $\mathbf{C}$ containing $\varphi_U(x)$, and has its maximum modulus there, the maximum modulus principle says that f must be a constant c in a neighbourhood of x. If X is connected, it follows that $f = c$ everywhere.

Though there are no holomorphic functions, there do exist meromorphic functions:

Definition 10 *A meromorphic function f on a Riemann surface X is a holomorphic map to the Riemann sphere $S = \mathbf{C} \cup \{\infty\}$.*

This means that if we remove $f^{-1}(\infty)$, then f is just a holomorphic function F with values in $\mathbf{C}$. If $f(x) = \infty$, and U is a coordinate neighbourhood of x, then using the coordinate z', $f\varphi_U^{-1}$ is holomorphic. But $\tilde{z} = 1/z$ if $z \neq 0$ which means that $(F \circ \varphi_U^{-1})^{-1}$ is holomorphic. Since it also vanishes,

$$F \circ \varphi_U^{-1} = \frac{a_0}{z^m} + \ldots$$

which is usually what we mean by a meromorphic function.

Example: A rational function

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials is a meromorphic function on the Riemann sphere S.

The definition above is a geometrical one – a map from one surface to another. On the other hand, if we think of it as a function with singularities, we can add and multiply – meromorphic functions form a field – which is the algebraic approach. These two viewpoints can be very valuable. The second one allows us to manipulate with ease: here is an example using the algebraic approach of a meromorphic function on the torus in Example 2.

Define

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum is over all non-zero $\omega = m\omega_1 + n\omega_2$. Since for $2|z| < |\omega|$

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| \leq 10\frac{|z|}{|\omega|^3}$$

this converges uniformly on compact sets so long as

$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3} < \infty.$$

But $m\omega_1 + n\omega_2$ is never zero if m, n are real so we have an estimate

$$|m\omega_1 + n\omega_2| \geq k\sqrt{m^2+n^2}$$

so by the integral test we have convergence. Because the sum is essentially over all equivalence classes

$$\wp(z + m\omega_1 + n\omega_2) = \wp(z)$$

so that this is a meromorphic function on the surface X. It is called the Weierstrass P-function.

It is a quite deep result that any closed Riemann surface has meromorphic functions. We are now going to consider them in more detail from the geometric point of view. So let

$$f : X \to S$$

be a meromorphic function. If the inverse image of $a \in S$ is infinite, then it has a limit point x by compactness of X. In a holomorphic coordinate around x with $z(x) = 0$, f is defined by a holomorphic function $F = f\varphi_U^{-1}$ with a sequence of points $z_n \to 0$ for which $F(z_n) - a = 0$. But the zeros of a holomorphic function are isolated, so we deduce that $f^{-1}(a)$ is a finite set. By a similar argument the points at which the derivative F' vanishes are finite in number (check using the chain rule that this condition is independent of the holomorphic coordinate). The points of X at which $F' = 0$ are called *ramification points.*

The word "ramification" means "branching". We defined it here analytically through the vanishing of a derivative, but we need to understand its geometric meaning. The simplest example is the map $f(z) = z^2$ from $\mathbf{C}$ to $\mathbf{C}$ so that $z = 0$ is a ramification point. In a neighbourhood of zero there is no single-valued inverse to f – in complex analysis we say that the square root $\sqrt{w}$ has two branches. The origin has one inverse image, any other point has two, but we can't distinguish between them because as w goes around a circle surrounding the origin one square root extends continuously to its negative. A similar phenomenon holds for the map $z \mapsto z^n$.

In fact, if f is any holomorphic function on $\mathbf{C}$ such that $f'(0) = 0$, we have

$$f(z) = z^n(a_0 + a_1 z + \ldots)$$

with $a_0 \neq 0$. We can expand

$$(a_0 + a_1 z + \ldots)^{1/n} = a_0^{1/n}(1 + b_1 z + \ldots)$$

in a power series and define

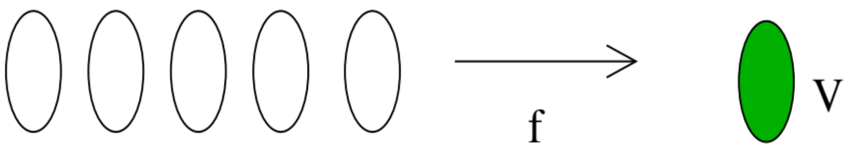
$$w = a_0^{1/n} z(1 + b_1 z + \ldots).$$

Since $w'(0) \neq 0$ we can think of w as a new coordinate and then the map becomes simply

$$w \mapsto w^n.$$

So, thinking geometrically of $\mathbf{C}$ as a Riemann surface where we are allowed to change coordinates, a ramification point is a map of the form $z \mapsto z^n$. The integer n is its multiplicity.

So now return to a holomorphic map $f : X \to S$. There are two types of points: if $F'(x) \neq 0$, then the inverse function theorem tells us that f maps a neighbourhood U_x of $x \in X$ homeomorphically to a neighbourhood V_x of $f(x) \in S$. Define V to be the intersection of the V_x as x runs over the finite set of points such that $f(x) = a$, then $f^{-1}V$ consists of a finite number d of open sets, each mapped homeomorphically onto V by f:



If $F'(x) = 0$ then the map looks like $w \mapsto w^n$. The inverse image of $f(x) = a$ is then a disjoint union of open sets, on some of which the map might map homeomorphicall to a disc, but where on at least one (containing x) the map is of the form $z \mapsto z^n$.

Removing the finite number of images under f of ramification points we get a sphere minus a finite number of points. This is connected. The number of points in the inverse image of a point in this punctured sphere is integer-valued and continuous, hence constant. It is called the *degree* d of the meromorphic function f.

With this we can determine the Euler characteristic of the Riemann surface S from the meromorphic function:

Theorem 3.2 *(Riemann-Hurwitz) Let $f : X \to S$ be a meromorphic function of degree d on a closed connected Riemann surface X, and suppose it has ramification points $x_1, \ldots, x_n$ where the local form of $f(x) - f(x_k)$ is a holomorphic function with a zero of multiplicity m_k. Then*

$$\chi(X) = 2d - \sum_{k=1}^{n}(m_k - 1)$$

Proof: The idea is to take a triangulation of the sphere S such that the image of the ramification points are vertices. This is straighforward. Now take a finite subcovering of S by open sets of the form V above where the map f is either a homeomorphism or of the form $z \mapsto z^m$. Subdivide the triangulation into smaller triangles such that each one is contained in one of the sets V. Then the inverse images of the vertices and edges of S form the vertices and edges of a triangulation of X.

If the triangulation of S has V vertices, E edges and F faces, then clearly the triangulation of X has dE edges and dF faces. It has fewer vertices, though — in a neighbourhood where f is of the form $w \mapsto w^m$ the origin is a single vertex instead of m of them. For each ramification point of order m_k we therefore have one vertex instead of m_k. The count of vertices is therefore

$$dV - \sum_{k=1}^{n}(m_k - 1).$$

Thus

$$\chi(X) = d(V - E + F) - \sum_{k=1}^{n}(m_k - 1) = 2d - \sum_{k=1}^{n}(m_k - 1)$$

using $\chi(S) = 2$. □

Clearly the argument works just the same for a holomorphic map $f : X \to Y$ and then

$$\chi(X) = d\chi(Y) - \sum_{k=1}^{n}(m_k - 1).$$

As an example, consider the Weierstrass P-function $\wp : T \to S$:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

We constructed this by adding and multiplying but we want to know geometrically what the map from a torus to a sphere looks like.

Firstly, $\wp$ has degree 2 since $\wp(z) = \infty$ only at $z = 0$ and there it has multiplicity 2. The multiplicity of a ramification point cannot be bigger than this beacuse then it will look like $z \mapsto z^n$ and a non-zero point will have at least n inverse images. Thus the only possible value at the ramification points here is $m_k = 2$. The Riemann-Hurwitz formula gives:

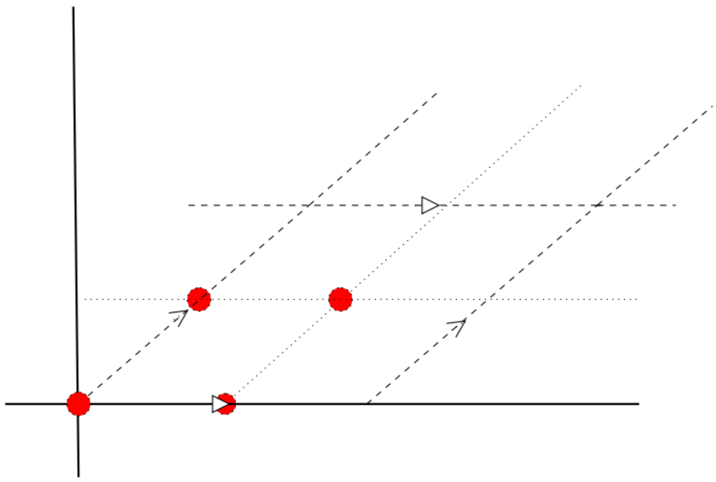
$$0 = 4 - n$$

so there must be exactly 4 ramification points. In fact we can see them directly, because $\wp(z)$ is an even function, so the derivative vanishes if $-z = z$. Of course at $z = 0$, $\wp(z) = \infty$ so we should use the other coordinate on S: $1/\wp$ has a zero of multiplicity 2 at $z = 0$. To find the other points recall that $\wp$ is doubly periodic so $\wp'$ vanishes where

$$z = -z + m\omega_1 + n\omega_2$$

for some integers m, n, and these are the four points

$$0, \omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2 :$$



The geometric Riemann-Hurwitz formula has helped us here in the analysis by showing us that the only zeros of $\wp'$ are the obvious ones.

A new look at the torus

Viewing the torus the points $z \in \mathbf{C}$ moduli $z \sim z + m\omega_1 + n\omega_2$ is not the only way to see it as a Riemann surface. We shall now look at it, via the P-function, in a manner which will show us how to construct other Riemann surfaces.

So consider again the P-function, thought of as a degree 2 map $\wp : T \to S$. It has 4 ramification points, whose images are ∞ and the three finite points e_1, e_2, e_3 where

$$e_1 = \wp(\omega_1/2), \quad e_2 = \wp(\omega_2/2), \quad e_3 = \wp((\omega_1 + \omega_2)/2).$$

So its derivative $\wp'(z)$ vanishes only at three points, each with multiplicity 1. At each of these points $\wp$ has the local form

$$\wp(z) = e_1 + (z - \omega_1/2)^2(a_0 + \ldots)$$

and so

$$\frac{1}{\wp'(z)^2}(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

is a well-defined holomorphic function on T away from $z = 0$. But $\wp(z) \sim z^{-2}$ near $z = 0$, and so $\wp'(z) \sim -2z^{-3}$ so this function is finite at $z = 0$ with value $1/4$. By the maximum argument, since T is compact, the function is a constant, namely $1/4$, and

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{1}$$

When we introduced the torus as Example 2, then z or $z + c$ were local holomorphic coordinates. But now if $z \neq 0$, $\wp(z)$ is finite and if $\wp'(z) \neq 0$ this gives us another local coordinate. From (1), $2\wp'\wp'' = 4q'(\wp)\wp'$ where $q(x) = (x - e_1)(x - e_2)(x - e_3)$ and so

$$\wp''(e_i) = \frac{1}{2}q'(e_i)$$

which is non-zero since e_1, e_2, e_3 are distinct. This means that $\wp'(z)$ is a local coordinate near $z = \omega_i/2$.

Finally, near $z = 0$,

$$\left(\frac{1}{\wp(z)}\right)' = -\frac{\wp'(z)}{\wp^2(z)} = 2z + \ldots$$

which means that $\wp'(z)/\wp^2(z)$ is a coordinate near $z = 0$.

We now see the torus rather differently. Consider

$$C = \{(x, y) \in \mathbf{C}^2 : y^2 = q(x) = 4(x - e_1)(x - e_2)(x - e_3)\}.$$

Now $\wp : T \to S$ is surjective (otherwise the degree would be zero!), so $\wp(z)$ takes every value in $\mathbf{C} = S \setminus \{\infty\}$. Moreover, since $\wp(-z) = \wp(z)$, we have $\wp'(-z) = -\wp'(z)$ hence for each value of x there is a value of z for which for $\wp'(z)$ takes each of the two values of y. Thus $(x, y) = (\wp(z), \wp'(z))$ defines a homeomorphism from $T \setminus \{0\}$ to C.

So we have $T = C \cup \{\infty\}$ with local coordinates:

- x near a if $q(a) \neq 0$
- y near a if $q(a) = 0$
- y/x^2 near ∞

Now generalize this to the case

$$C = \{(x, y) \in \mathbf{C}^2 : y^2 = q(x) = 4(x - e_1)(x - e_2) \dots (x - e_n)\}$$

where the e_i are distinct. As above we define a local coordinate x near a if $q(a) \neq 0$. Then

$$y = \pm \exp \left[\frac{1}{2} \int_c^x \frac{q'(z)}{q(z)} dz \right]$$

defines y as a locally invertible holomorphic function of x.

Near $x = e_i$, $q'(x) \neq 0$ so by the inverse function theorem $x = f(q(x))$ and $x = f(y^2)$ is holomorphic, so y is a local coordinate.

Near $x = \infty$ put $z = 1/x$ and if $n = 2m$, $w = y/x^m$ then

$$w^2 = (1 - e_1 z)(1 - e_2 z) \dots (1 - e_{2m} z).$$

Since $z = 0$ is not a root of the polynomial, this is like the first case above and z is a coordinate: we have then defined a Riemann surface structure on

$$X = C \cup \{\infty, -\infty\}$$

where the two extra points are given by $(w = \pm 1, z = 0)$.

If $n = 2m + 1$, put $w = y/x^{m+1}$ and then

$$w^2 = z(1 - ze_1) \dots (1 - ze_{2m+1})$$

and now since $w = 0$ is a root, we use w as a coordinate and define a Riemann surface structure on

$$X = C \cup \{\infty\}$$

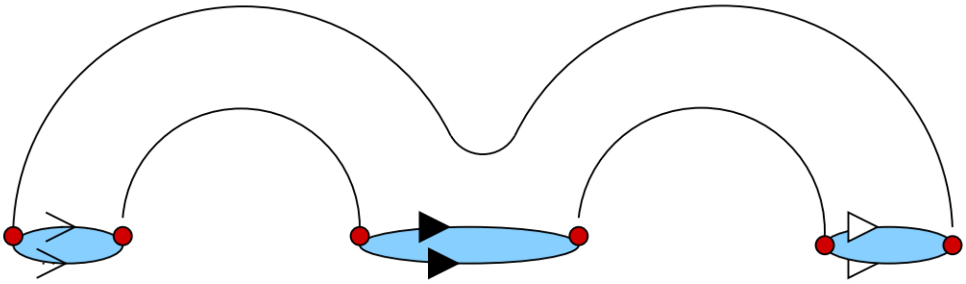
where the single extra point is given by $(w = 0, z = 0)$.

The Riemann-Hurwitz formula enables us to calculate the Euler characteristic of X. We can use x as a meromorphic function.

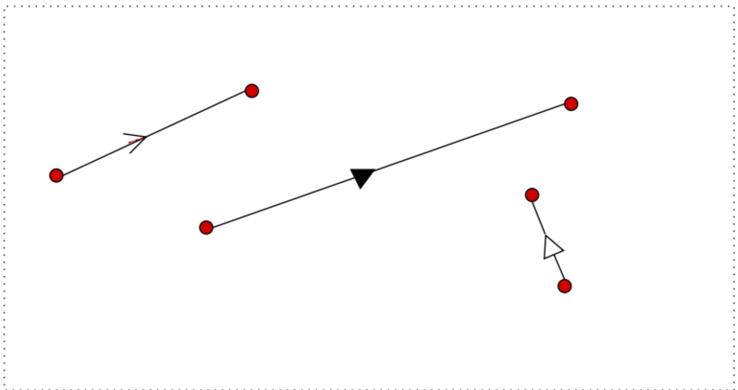
Firstly, given x, $y^2 = q(x)$ has two solutions in general so the degree of the map is two. The ramification points are where $x' = 0$ but since x is a coordinate where

$q(x) \neq 0$ this can only occur at $x = e_i$ or $x = \infty$. At $x = e_i$, y is a coordinate and $y = 0$. Since $x = f(y^2)$, we see that $x' = 0$. Because the map is of degree two the multiplicty can only be two. At infinity we have a ramification point if n is odd and not if n is even. Thus, applying Riemann-Hurwitz $\chi(X) = 4 - n$ if n is even and $4 - (n+1)$ if n is odd.

This type of Riemann surface is called *hyperelliptic*. Since the two values of $y = \sqrt{q(x)}$ only differ by a sign, we can think of $(y, x) \mapsto (-y, x)$ as being a holomorphic homeomorphism from X to X, and then x is a coordinate on the space of orbits. Topologically we can cut the surface in two – an "upper" and "lower" half – and identify on the points on the boundary to get a sphere:



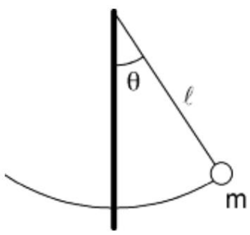
It is common also to view this downstairs on the Riemann sphere and insert cuts between pairs of zeros of the polynomial $p(z)$:



Hyperelliptic Riemann surfaces occcur in a number of dynamical problems where one needs to integrate

$$\int \frac{du}{\sqrt{(u - e_1)(u - e_2)\dots(u - e_n)}}.$$

The simplest example is the pendulum:



$$\theta'' = -(g/\ell)\sin\theta$$

which integrates once to

$$\theta'^2 = 2(g/\ell)\cos\theta + c.$$

Substituting $v = e^{i\theta}$ we get

$$v' = i\sqrt{2(g/\ell)(v^3 + v) + cv^2}.$$

By changing variables this can be brought into this form

$$2cdt = \frac{dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}$$

which can be solved with $x = \wp(ct)$. So time becomes (the real part of) the parameter z on $\mathbf{C}$. In the torus this is a circle, so (no surprise here!) the solutions to the pendulum equation are periodic.

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