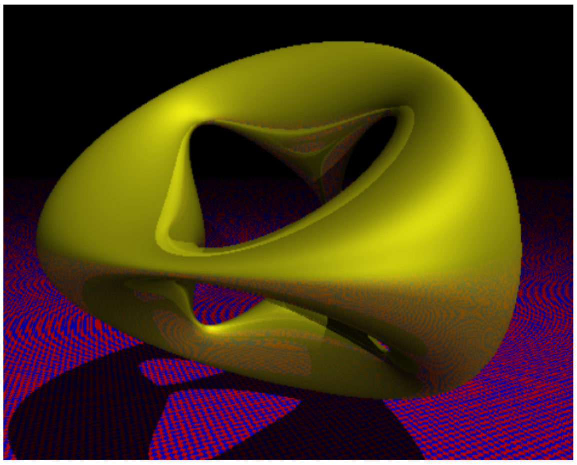
Surfaces in $\mathbf{R}^3$

Definitions

At this point we return to surfaces embedded in Euclidean space, and consider the differential geometry of these:



We shall not forget the idea of an abstract surface though, and as we meet objects which we call *intrinsic* we shall show how to define them on a surface which is not sitting in $\mathbf{R}^3$. These remarks are printed in a smaller typeface.

Definition 11 *A smooth surface in $\mathbf{R}^3$ is a subset $X \subset \mathbf{R}^3$ such that each point has a neighbourhood $U \subset X$ and a map $\mathbf{r} : V \to \mathbf{R}^3$ from an open set $V \subseteq \mathbf{R}^2$ such that*

- $\mathbf{r} : V \to U$ *is a homeomorphism*
- $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ *has derivatives of all orders*
- *at each point* $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ *and* $\mathbf{r}_v = \partial\mathbf{r}/\partial v$ *are linearly independent.*

Already in the definition we see that X is a topological surface as in Definition 2, since $\mathbf{r}$ defines a homeomorphism $\varphi_U : U \to V$. The last two conditions make sense if we use the *implicit function theorem* (see Appendix 1). This tells us that a local invertible change of variables in $\mathbf{R}^3$ "straightens out" the surface: it can be locally defined by $x_3 = 0$ where (x_1, x_2, x_3) are (nonlinear) local coordinates on $\mathbf{R}^3$. For any two open sets U, U', we get a smooth invertible map from an open set of $\mathbf{R}^3$ to another which takes $x_3 = 0$ to $x'_3 = 0$. This means that each map $\varphi_{U'}\varphi_U^{-1}$ is a smooth invertible homeomorphism. This motivates the definition of an abstract smooth surface:

Definition 12 *A smooth surface is a surface with a class of homeomorphisms* φ_U *such that each map* $\varphi_{U'}\varphi_U^{-1}$ *is a smoothly invertible homeomorphism.*

Clearly, since a holomorphic function has partial derivatives of all orders in x, y, a Riemann surface is an example of an abstract smooth surface. Similarly, we have

Definition 13 *A smooth map between smooth surfaces* X *and* Y *is a continuous map* $f : X \to Y$ *such that for each smooth coordinate system* φ_U *on* U *containing* x *on* X *and* ψ_W *defined in a neighbourhood of* $f(x)$ *on* Y*, the composition*

$$\psi_W \circ f \circ \varphi_U^{-1}$$

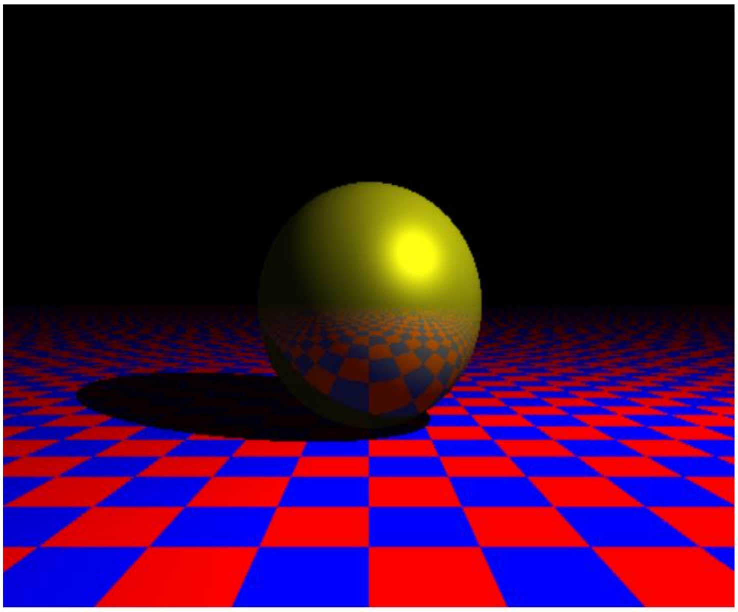
is smooth.

We now return to surfaces in $\mathbf{R}^3$:

Examples:

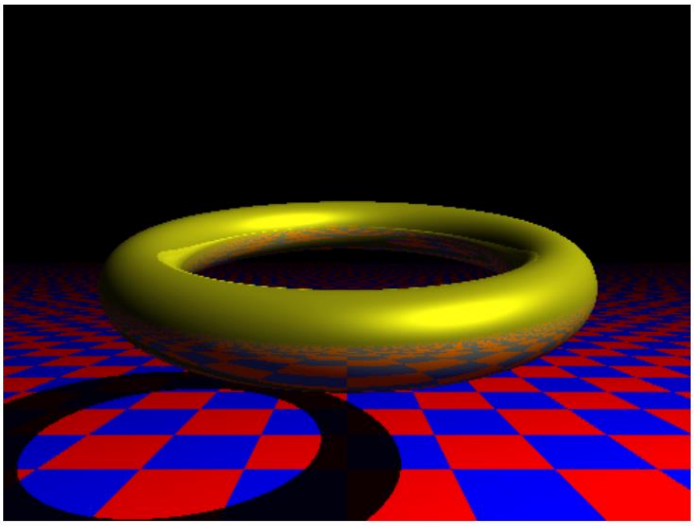
1) A sphere:

$$\mathbf{r}(u, v) = a \sin u \sin v\, \mathbf{i} + a \cos u \sin v\, \mathbf{j} + a \cos v\, \mathbf{k}$$



2) A torus:

$$\mathbf{r}(u, v) = (a + b \cos u)(\cos v\, \mathbf{i} + \sin v\, \mathbf{j}) + b \sin u\, \mathbf{k}$$



These are the only compact surfaces it is easy to write down, but the following non-compact ones are good for local discussions:

Examples:

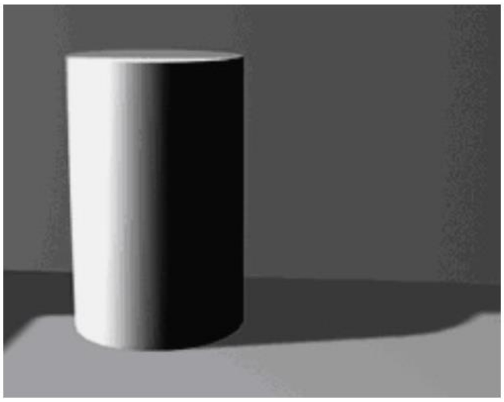
1) A plane:

$$\mathbf{r}(u,v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

for constant vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ where $\mathbf{b}, \mathbf{c}$ are linearly independent.

2) A cylinder:

$$\mathbf{r}(u,v) = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$

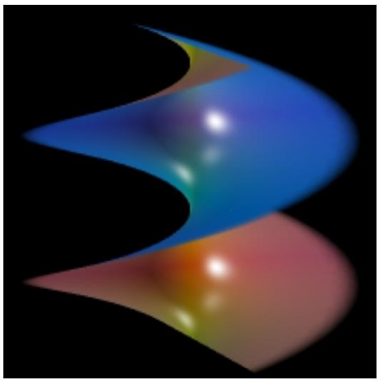


3) A cone:

$$\mathbf{r}(u,v) = au\cos v\,\mathbf{i} + au\sin v\,\mathbf{j} + u\mathbf{k}$$

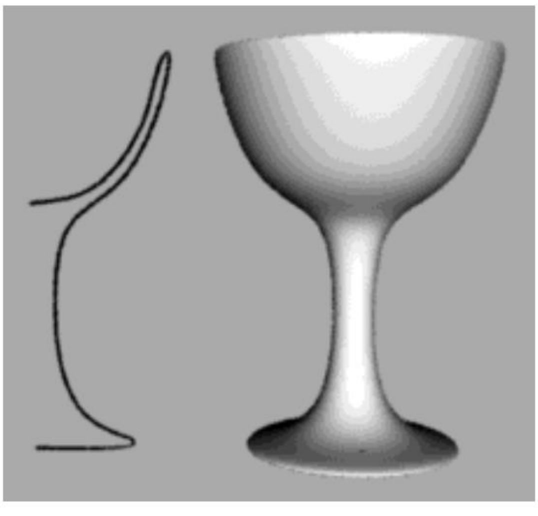
4) A helicoid:

$$\mathbf{r}(u,v) = au\cos v\,\mathbf{i} + au\sin v\,\mathbf{j} + v\mathbf{k}$$



5) A surface of revolution:

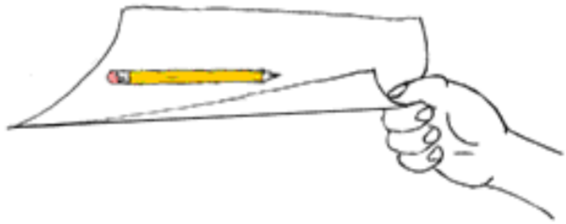
$$\mathbf{r}(u,v) = f(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$



6) A developable surface: take a curve $\boldsymbol{\gamma}(u)$ parametrized by arc length and set

$$\mathbf{r}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\gamma}'(u)$$

This is the surface formed by bending a piece of paper:



A change of parametrization of a surface is the composition

$$\mathbf{r} \circ f : V' \to \mathbf{R}^3$$

where $f : V' \to V$ is a *diffeomorphism* – an invertible map such that f and f^{-1} have derivatives of all orders. Note that if

$$f(x,y) = (u(x,y), v(x,y))$$

then by the chain rule

$$\begin{aligned} (\mathbf{r} \circ f)_x &= \mathbf{r}_u u_x + \mathbf{r}_v v_x \\ (\mathbf{r} \circ f)_y &= \mathbf{r}_u u_y + \mathbf{r}_v v_y \end{aligned}$$

so

$$\begin{pmatrix} (\mathbf{r} \circ f)_x \\ (\mathbf{r} \circ f)_y \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix}.$$

Since f has a differentiable inverse, the Jacobian matrix is invertible, so $(\mathbf{r} \circ f)_x$ and $(\mathbf{r} \circ f)_y$ are linearly independent if $\mathbf{r}_u, \mathbf{r}_v$ are.

Example: The (x,y) plane

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j}$$

has a different parametrization in polar coordinates

$$\mathbf{r} \circ f(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}.$$

We have to consider changes of parametrizations when we pass from one open set V to a neighbouring one V'.

Definition 14 *The tangent plane (or* tangent space*) of a surface at the point a is the vector space spanned by $\mathbf{r}_u(a), \mathbf{r}_v(a)$.*

Note that this space is independent of parametrization. One should think of the origin of the vector space as the point a.

Definition 15 *The vectors*

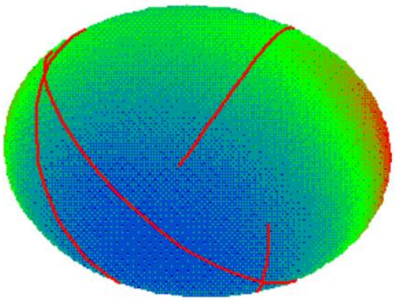
$$\pm\frac{\mathbf{r}_u\wedge\mathbf{r}_v}{|\mathbf{r}_u\wedge\mathbf{r}_v|}$$

are the two unit normals ("inward and outward") to the surface at (u, v).

The first fundamental form

Definition 16 *A smooth curve lying in the surface is a map* $t \mapsto (u(t), v(t))$ *with derivatives of all orders such that* $\boldsymbol{\gamma}(t) = \mathbf{r}(u(t), v(t))$ *is a parametrized curve in* $\mathbf{R}^3$.

A parametrized curve means that $u(t), v(t)$ have derivatives of all orders and $\boldsymbol{\gamma}' = \mathbf{r}_u u' + \mathbf{r}_v v' \neq 0$. The definition of a surface implies that $\mathbf{r}_u, \mathbf{r}_v$ are linearly independent, so this condition is equivalent to $(u', v') \neq 0$.



The arc length of such a curve from $t = a$ to $t = b$ is:

$$\begin{aligned}\int_a^b |\boldsymbol{\gamma}'(t)|dt &= \int_a^b \sqrt{\boldsymbol{\gamma}'\cdot\boldsymbol{\gamma}'}dt \\ &= \int_a^b \sqrt{(\mathbf{r}_u u' + \mathbf{r}_v v')\cdot(\mathbf{r}_u u' + \mathbf{r}_v v')}dt \\ &= \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt\end{aligned}$$

where

$$E = \mathbf{r}_u\cdot\mathbf{r}_u, \quad F = \mathbf{r}_u\cdot\mathbf{r}_v, \quad F = \mathbf{r}_v\cdot\mathbf{r}_v.$$

Definition 17 *The first fundamental form of a surface in* $\mathbf{R}^3$ *is the expression*

$$Edu^2 + 2Fdudv + Gdv^2$$

where $E = \mathbf{r}_u\cdot\mathbf{r}_u, F = \mathbf{r}_u\cdot\mathbf{r}_v, G = \mathbf{r}_v\cdot\mathbf{r}_v$.

The first fundamental form is just the quadratic form

$$Q(\mathbf{v}, \mathbf{v}) = \mathbf{v} \cdot \mathbf{v}$$

on the tangent space written in terms of the basis $\mathbf{r}_u, \mathbf{r}_v$. It is represented in this basis by the symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

So why do we write it as $Edu^2 + 2Fdudv + Gdv^2$? At this stage it is not worth worrying about what exactly du^2 is, instead let's see how the terminology helps to manipulate the formulas.

For example, to find the length of a curve $u(t), v(t)$ on the surface, we calculate

$$\int \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt$$

– divide the first fundamental form by dt^2 and multiply its square root by dt.

Furthermore if we change the parametrization of the surface via $u(x, y), v(x, y)$ and try to find the length of the curve $(x(t), y(t))$ then from first principles we would calculate

$$u' = u_x x' + u_y y' \qquad v' = v_x x' + v_y y'$$

by the chain rule and then

$$\begin{aligned} Eu'^2 + 2Fu'v' + Gv'^2 &= E(u_x x' + u_y y')^2 + 2F(u_x x' + u_y y')(v_x x' + \ldots \\ &= (Eu_x^2 + 2Fu_x v_x + Gv_x^2)x'^2 + \ldots \end{aligned}$$

which is heavy going. Instead, using du, dv etc. we just write

$$\begin{aligned} du &= u_x dx + u_y dy \\ dv &= v_x dx + v_y dy \end{aligned}$$

and substitute in $Edu^2 + 2Fdudv + Gdv^2$ to get $E'dx^2 + 2F'dxdy + G'dy^2$. Using matrices, we can write this transformation as

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix}$$

Example: For the plane

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j}$$

we have $\mathbf{r}_x = \mathbf{i}, \mathbf{r}_y = \mathbf{j}$ and so the first fundamental form is

$$dx^2 + dy^2.$$

Now change to polar coordinates $x = r\cos\theta, y = r\sin\theta$. We have

$$
\begin{aligned}
dx &= dr\cos\theta - r\sin\theta d\theta \\
dy &= dr\sin\theta + r\cos\theta d\theta
\end{aligned}
$$

so that

$$dx^2 + dy^2 = (dr\cos\theta - r\sin\theta d\theta)^2 + (dr\sin\theta + r\cos\theta d\theta)^2 = dr^2 + r^2 d\theta^2$$

Here are some examples of first fundamental forms:

Examples:

1. The cylinder

$$\mathbf{r}(u,v) = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}.$$

We get

$$\mathbf{r}_u = \mathbf{k}, \quad \mathbf{r}_v = a(-\sin v\,\mathbf{i} + \cos v\,\mathbf{j})$$

so

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2$$

giving

$$\boxed{du^2 + a^2 dv^2}$$

2. The cone

$$\mathbf{r}(u,v) = a(u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j}) + u\mathbf{k}.$$

Here

$$\mathbf{r}_u = a(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + \mathbf{k}, \quad \mathbf{r}_v = a(-u\sin v\,\mathbf{i} + u\cos v)$$

so

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + a^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2u^2$$

giving

$$\boxed{(1+a^2)du^2 + a^2u^2dv^2}$$

3. The sphere

$$\mathbf{r}(u,v) = a\sin u\sin v\,\mathbf{i} + a\cos u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$$

gives

$$\mathbf{r}_u = a\cos u\sin v\,\mathbf{i} - a\sin u\sin v\,\mathbf{j}, \quad \mathbf{r}_v = a\sin u\cos v\,\mathbf{i} + a\cos u\cos v\,\mathbf{j} - a\sin v\,\mathbf{k}$$

so that

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = a^2\sin^2 v, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2$$

and so we get the first fundamental form

$$\boxed{a^2dv^2 + a^2\sin^2 v du^2}$$

4. A surface of revolution

$$\mathbf{r}(u,v) = f(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$$

has

$$\mathbf{r}_u = f'(u)(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + \mathbf{k}, \quad \mathbf{r}_v = f(u)(-\sin v\,\mathbf{i} + \cos v\,\mathbf{j})$$

so that

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + f'(u)^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = f(u)^2$$

gives

$$\boxed{(1 + f(u)'^2)du^2 + f(u)^2dv^2}$$

5. A developable surface

$$\mathbf{r}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{t}(u).$$

here the curve is parametrized by arc length $u = s$ so that

$$\mathbf{r}_u = \mathbf{t}(u) + v\mathbf{t}'(u) = \mathbf{t} + v\kappa\mathbf{n}, \quad \mathbf{r}_v = \mathbf{t}$$

where $\mathbf{n}$ is the normal to the curve and κ its curvature. This gives

$$\boxed{(1 + v^2\kappa^2)du^2 + 2dudv + dv^2}$$

The analogue of the first fundamental form on an abstract smooth surface X is called a *Riemannian metric*. On each open set U with coordinates (u,v) we ask for smooth functions E, F, G with $E > 0, G > 0, EG - F^2 > 0$ and on an overlapping neighbourhood with coordinates (x,y) smooth functions E', F', G' with the same properties and the transformation law:

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix}$$

A smooth curve on X is defined to be a map $\gamma : [a,b] \to X$ such that $\varphi_U \gamma$ is smooth for each coordinate neighbourhood U on the image. The length of such a curve is well-defined by a Riemannian metric.

Examples:

1. The torus as a Riemann surface has the metric

$$dzd\bar{z} = dx^2 + dy^2$$

as the local holomorphic coordinates are z and $z + m\omega_1 + n\omega_2$ so that the Jacobian matrix is the identity. We could also multiply this by any positive smooth doubly-periodic function.

2. The hyperelliptic Riemann surface $w^2 = p(z)$ where $p(z)$ is of degree $2m$ has Riemannian metrics given by

$$\frac{1}{|w|^2}(a_0 + a_1|z|^2 + \ldots + a_{m-2}|z|^{2(m-2)})dzd\bar{z}$$

where the a_i are positive constants.

3. The upper half-space $\{x + iy \in \mathbf{C} : y > 0\}$ has the metric

$$\frac{dx^2 + dy^2}{y^2}.$$

None of these have anything to do with the first fundamental form of the surface embedded in $\mathbf{R}^3$.

We introduced the first fundamental form to measure lengths of curves on a surface but it does more besides. Firstly if two curves $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ on the surface intersect, the angle θ between them is given by

$$\cos\theta = \frac{\boldsymbol{\gamma}_1' \cdot \boldsymbol{\gamma}_2'}{|\boldsymbol{\gamma}_1'||\boldsymbol{\gamma}_2'|} \tag{2}$$

But $\boldsymbol{\gamma}_i' = \mathbf{r}_u u_i' + \mathbf{r}_v v_i'$ so

$$\begin{aligned} \boldsymbol{\gamma}_i' \cdot \boldsymbol{\gamma}_j' &= (\mathbf{r}_u u_i' + \mathbf{r}_v v_i') \cdot (\mathbf{r}_u u_j' + \mathbf{r}_v v_j') \\ &= E u_i' u_j' + F(u_i' v_j' + u_j' v_i') + G v_i' v_j' \end{aligned}$$

and each term in (2) can be expressed in terms of the curves and the coefficients of the first fundamental form.

We can also define *area* using the first fundamental form:

Definition 18 *The area of the domain $\mathbf{r}(U) \subset \mathbf{R}^3$ in a surface is defined by*

$$\int_U |\mathbf{r}_u \wedge \mathbf{r}_v| dudv = \int_U \sqrt{EG - F^2} dudv.$$

The second form of the formula comes from the identity

$$|\mathbf{r}_u \wedge \mathbf{r}_v|^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = EG - F^2.$$

Note that the definition of area is independent of parametrization for if

$$\mathbf{r}_x = \mathbf{r}_u u_x + \mathbf{r}_v v_x, \quad \mathbf{r}_y = \mathbf{r}_u u_y + \mathbf{r}_v v_y$$

then

$$\mathbf{r}_x \wedge \mathbf{r}_y = (u_x v_y - v_x u_y)\mathbf{r}_u \wedge \mathbf{r}_v$$

so that

$$\int_U |\mathbf{r}_x \wedge \mathbf{r}_y| dx dy = \int_U |\mathbf{r}_u \wedge \mathbf{r}_v||u_x v_y - v_x u_y| dx dy = \int_U |\mathbf{r}_u \wedge \mathbf{r}_v| du dv$$

using the formula for change of variables in multiple integration.

Example: Consider a surface of revolution

$$(1 + f'(u)^2)du^2 + f(u)^2 dv^2$$

and the area between $u = a, u = b$. We have

$$EG - F^2 = f(u)^2(1 + f'(u)^2)$$

so the area is

$$\int_a^b f(u)\sqrt{1 + f'(u)^2} du dv = 2\pi \int_a^b f(u)\sqrt{1 + f'(u)^2} du.$$

If a closed surface X is triangulated so that each face lies in a coordinate neighbourhood, then we can define the area of X as the sum of the areas of the faces by the formula above. It is independent of the choice of triangulation.

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