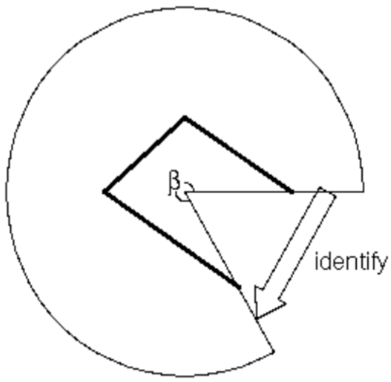
Isometric surfaces

Definition 19 *Two surfaces X, X' are isometric if there is a smooth homeomorphism $f : X \to X'$ which maps curves in X to curves in X' of the same length.*

A practical example of this is to take a piece of paper and bend it: the lengths of curves in the paper do not change. The cone and a subset of the plane are isometric this way:



Analytically this is how to tell if two surfaces are isometric:

Theorem 4.1 *The coordinate patches of surfaces U and U' are isometric if and only if there exist parametrizations $\mathbf{r} : V \to \mathbf{R}^3$ and $\mathbf{r}' : V \to \mathbf{R}^3$ with the same first fundamental form.*

Proof: Suppose such a parametrization exists, then the identity map is an isometry since the first fundamental form determines the length of curves.

Conversely, suppose X, X' are isometric using the function $f : V \to V'$. Then

$$\mathbf{r}' \circ f : V \to \mathbf{R}^3, \qquad \mathbf{r} : V \to \mathbf{R}^3$$

are parametrizations using the same open set V, so the first fundamental forms are

$$\tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2, \quad Edu^2 + 2Fdudv + Gdv^2$$

and since f is an isometry

$$\int_I \sqrt{\tilde{E}u'^2 + 2\tilde{F}u'v' + \tilde{G}v'^2}dt = \int_I \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt$$

for all curves $t \mapsto (u(t), v(t))$ and *all intervals*. Since

$$\frac{d}{dt} \int_a^{a+t} h(u) du = h(a)$$

this means that

$$\sqrt{\tilde{E} u'^2 + 2\tilde{F} u' v' + \tilde{G} v'^2} = \sqrt{E u'^2 + 2F u' v' + G v'^2}$$

for all $u(t), v(t)$. So, choosing u, v appropriately:

$$\begin{aligned} u = t, v = a &\Rightarrow \tilde{E} = E \\ u = a, v = t &\Rightarrow \tilde{G} = G \\ u = t, v = t &\Rightarrow \tilde{F} = F \end{aligned}$$

and we have the same first fundamental form as required.

Example:

The cone has first fundamental form

$$(1 + a^2) du^2 + a^2 u^2 dv^2.$$

Put

$$r = \sqrt{1 + a^2} u$$

then we get

$$dr^2 + \left(\frac{a^2}{1 + a^2} \right) r^2 dv^2$$

and now put

$$\theta = \sqrt{\frac{a^2}{1 + a^2}} v$$

to get the plane in polar coordinates

$$dr^2 + r^2 d\theta^2.$$

Note that as $0 \leq v \leq 2\pi$, $0 \leq \theta \leq \beta$ where

$$\beta = \sqrt{\frac{a^2}{1 + a^2}} 2\pi < 2\pi$$

as in the picture.

Example: Consider the unit disc $D = \{x + iy \in \mathbf{C} | x^2 + y^2 < 1\}$ with first fundamental form

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

and the upper half plane $H = \{u + iv \in \mathbf{C} | v > 0\}$ with the first fundamental form

$$\frac{du^2 + dv^2}{v^2}.$$

We shall show that there is an isometry from H to D given by

$$w \mapsto z = \frac{w - i}{w + i}$$

where $w = u + iv \in H$ and $z = x + iy \in D$.

We write $|dz|^2 = dx^2 + dy^2$ and $|dw|^2 = du^2 + dv^2$. If $w = f(z)$ where $f : D \to H$ is holomorphic then

$$f'(z) = u_x + iv_x = v_y - iu_y$$

and so

$$|f'(z)|^2 |dz|^2 = (u_x^2 + v_x^2)(dx^2 + dy^2) = (u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2 = du^2 + dv^2 = |dw|^2.$$

Thus we can substitute

$$|dw|^2 = \left|\frac{dw}{dz}\right|^2 |dz|^2 \tag{3}$$

to calculate how the first fundamental form is transformed by such a map.

The Möbius transformation

$$w \mapsto z = \frac{w - i}{w + i} \tag{4}$$

restricts to a smooth bijection from H to D because $w \in H$ if and only if $|w - i| < |w + i|$, and its inverse is also a Möbius transformation and hence is also smooth. Substituting (4) and (3) with

$$\frac{dw}{dz} = \frac{1}{w + i} - \frac{(w - i)}{(w + i)^2} = \frac{2i}{(w + i)^2}$$

into $v^{-2}|dw|^2$ gives $4(1 - |z|^2)^{-2}|dz|^2$, so this Möbius transformation gives us an isometry from H to D as required.

The second fundamental form

The first fundamental form describes the intrinsic geometry of a surface – the experience of an insect crawling around it. It is this that we can generalize to abstract surfaces. The second fundamental form relates to the way the surface sits in $\mathbf{R}^3$, though as we shall see, it is not independent of the first fundamental form.

First take a surface $\mathbf{r}(u, v)$ and push it inwards a distance t along its normal to get a one-parameter family of surfaces:

$$\mathbf{R}(u, v, t) = \mathbf{r}(u, v) - t\mathbf{n}(u, v)$$

with

$$\mathbf{R}_u = \mathbf{r}_u - t\mathbf{n}_u, \quad \mathbf{R}_v = \mathbf{r}_v - t\mathbf{n}_v.$$

We now have a first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ depending on t and we calculate

$$\frac{1}{2}\frac{\partial}{\partial t}(Edu^2 + 2Fdudv + Gdv^2)|_{t=0} = -(\mathbf{r}_u \cdot \mathbf{n}_u du^2 + (\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u)dudv + \mathbf{r}_v \cdot \mathbf{n}_v dv^2).$$

The right hand side is the second fundamental form. From this point of view it is clearly the same type of object as the first fundamental form — a quadratic form on the tangent space.

In fact it is useful to give a slightly different expression. Since $\mathbf{n}$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_v$,

$$0 = (\mathbf{r}_u \cdot \mathbf{n})_u = \mathbf{r}_{uu} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_u$$

and similarly

$$\mathbf{r}_{uv} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_v = 0, \quad \mathbf{r}_{vu} \cdot \mathbf{n} + \mathbf{r}_v \cdot \mathbf{n}_u = 0$$

and since $\mathbf{r}_{uv} = \mathbf{r}_{vu}$ we have $\mathbf{r}_u \cdot \mathbf{n}_v = \mathbf{r}_v \cdot \mathbf{n}_u$. We then define:

Definition 20 *The second fundamental form of a surface is the expression*

$$Ldu^2 + 2Mdudv + Ndv^2$$

where $L = \mathbf{r}_{uu} \cdot \mathbf{n}$, $M = \mathbf{r}_{uv} \cdot \mathbf{n}$, $N = \mathbf{r}_{vv} \cdot \mathbf{n}$.

Examples:

1) The plane

$$\mathbf{r}(u, v) = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$$

has $\mathbf{r}_{uu} = \mathbf{r}_{uv} = \mathbf{r}_{vv} = 0$ so the second fundamental form vanishes.

2) The sphere of radius a: here with the origin at the centre, $\mathbf{r} = a\mathbf{n}$ so

$$\mathbf{r}_u \cdot \mathbf{n}_u = a^{-1}\mathbf{r}_u \cdot \mathbf{r}_u, \quad \mathbf{r}_u \cdot \mathbf{n}_v = a^{-1}\mathbf{r}_u \cdot \mathbf{r}_v, \quad \mathbf{r}_v \cdot \mathbf{n}_v = a^{-1}\mathbf{r}_v \cdot \mathbf{r}_v$$

and

$$Ldu^2 + 2Mdudv + Ndv^2 = a^{-1}(Edu^2 + 2Fdudv + Gdv^2).$$

The plane is characterised by the vanishing of the second fundamental form:

Proposition 4.2 *If the second fundamental form of a surface vanishes, it is part of a plane.*

Proof: If the second fundamental form vanishes,

$$\mathbf{r}_u \cdot \mathbf{n}_u = 0 = \mathbf{r}_v \cdot \mathbf{n}_u = \mathbf{r}_u \cdot \mathbf{n}_v = \mathbf{r}_v \cdot \mathbf{n}_v$$

so that

$$\mathbf{n}_u = \mathbf{n}_v = 0$$

since $\mathbf{n}_u, \mathbf{n}_v$ are orthogonal to $\mathbf{n}$ and hence linear combinations of $\mathbf{r}_u, \mathbf{r}_v$. Thus $\mathbf{n}$ is constant. This means

$$(\mathbf{r} \cdot \mathbf{n})_u = \mathbf{r}_u \cdot \mathbf{n} = 0, \quad (\mathbf{r} \cdot \mathbf{n})_v = \mathbf{r}_v \cdot \mathbf{n} = 0$$

and so

$$\mathbf{r} \cdot \mathbf{n} = const$$

which is the equation of a plane. □

Consider now a surface given as the graph of a function $z = f(x, y)$:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

Here

$$\mathbf{r}_x = \mathbf{i} + f_x\mathbf{k}, \qquad \mathbf{r}_y = \mathbf{j} + f_y\mathbf{k}$$

and so

$$\mathbf{r}_{xx} = f_{xx}\mathbf{k}, \quad \mathbf{r}_{xy} = f_{xy}\mathbf{k}, \quad \mathbf{r}_{yy} = f_{yy}\mathbf{k}.$$

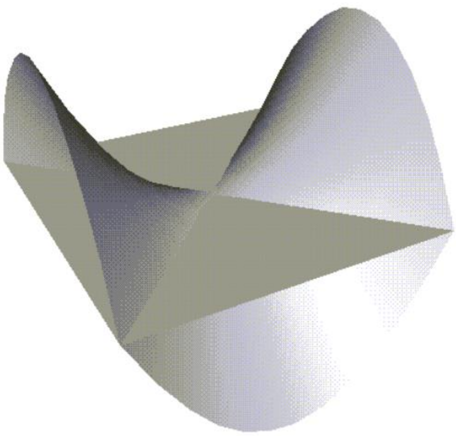
At a critical point of f, $f_x = f_y = 0$ and so the normal is $\mathbf{k}$. The second fundamental form is then the *Hessian* of the function at this point:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

We can use this to qualitatively describe the behaviour of the second fundamental form at different points on the surface. For any point P parametrize the surface by its projection on the tangent plane and then $f(x, y)$ is the height above the plane. Now use the theory of critical points of functions of two variables.

If $f_{xx}f_{yy} - f_{xy}^2 > 0$ then the critical point is a local maximum if the matrix is negative definite and a local minimum if it is positive definite. For the surface the difference is only in the choice of normal so the local picture of the surface is like the sphere – it lies on one side of the tangent plane at the point P.

If on the other hand $f_{xx}f_{yy} - f_{xy}^2 < 0$ we have a saddle point and the surface lies on both sides of the tangent plane:



In fact any closed surface X in $\mathbf{R}^3$, not just rabbit-shaped ones, must have points of the first type.

Proposition 4.3 *Any closed surface X in $\mathbf{R}^3$ has points at which the second fundamental form is negative definite.*

Proof: Since X is compact, it is bounded and so can be surrounded by a large sphere centre the origin. Gradually deflate the sphere until at radius R it touches X at a point. Let this be the direction of the z-axis and describe X locally as the graph of a function f as above. Then X lies below the sphere so

$$f - \sqrt{R^2 - x^2 - y^2} \le 0$$

with $f(0) = R$ and $f_x(0) = f_y(0) = 0$. Hence

$$\frac{1}{2}(f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2) + \frac{1}{2R}(x^2 + y^2) \le 0$$

so

$$Lx^2 + 2Mxy + Ny^2 \le -\frac{1}{R}(x^2 + y^2).$$

□

It is easy to understand qualitatively the behaviour of a surface from whether $LN - M^2$ is positive or not. In fact there is a closely related function called the Gaussian curvature which we shall study next.

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