# The Gaussian curvature

**Definition 21** *The Gaussian curvature of a surface in* $\mathbf{R}^3$ *is the function*

$$K = \frac{LN - M^2}{EG - F^2}$$

Note that under a coordinate change

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix}$$

so taking determinants

$$(u_x v_y - u_y v_x)^2 (EG - F^2) = (E'G' - F'^2).$$

Since the second fundamental form is a quadratic form on the tangent space just like the first, it undergoes the same transformation, so the ratio $(LN - M^2)/(EG - F^2)$ is independent of the choice of coordinates.

**Examples:**

1. For a plane, $L = M = N = 0$ so $K = 0$
2. For a sphere of radius $a$, the second fundamental form is $a^{-1}$ times the first so that $K = a^{-2}$.

We defined $K$ in terms of the second fundamental form which we said describes the extrinsic geometry of the surface. In fact it only depends on $E, F, G$ and its derivatives, and so is intrinsic – our insect crawling on the surface could in principle work it out. It was Gauss who showed this in 1828, a result he was particularly pleased with.

What it means is that if two surfaces are locally isometric, then the isometry maps the Gaussian curvature of one to the Gaussian curvature of the other – for example the Gaussian curvature of a bent piece of paper is zero because it is isometric to the plane. Also, we can define Gaussian curvature for an abstract Riemannian surface.

We prove Gauss's "egregious theorem", as he proudly called it, by a calculation. We consider locally a smooth family of tangent vectors

$$\mathbf{a} = f\mathbf{r}_u + g\mathbf{r}_v$$

where $f$ and $g$ are functions of $u, v$. If we differentiate with respect to $u$ or $v$ this is no longer necessarily tangential, but we can remove its normal component to make it so, and call this the *tangential derivative*:

$$\begin{aligned} \nabla_u \mathbf{a} &= \mathbf{a}_u - (\mathbf{n} \cdot \mathbf{a}_u)\mathbf{n} \\ &= \mathbf{a}_u + (\mathbf{n}_u \cdot \mathbf{a})\mathbf{n} \end{aligned}$$

since **a** and **n** are orthogonal.

The important thing to note is that this tangential derivative only depends on $E, F, G$ and their derivatives, because we are taking a tangent vector like $\mathbf{r}_u$, differentiating it to get $\mathbf{r}_{uu}$ and $\mathbf{r}_{uv}$ and then projecting back onto the tangent plane which involves taking dot products like $\mathbf{r}_{uu} \cdot \mathbf{r}_u = (\mathbf{r}_u \cdot \mathbf{r}_u)_u/2 = E_u/2$ etc.

Now differentiate $\nabla_u \mathbf{a}$ tangentially with respect to $v$:

$$\nabla_v \nabla_u \mathbf{a} = \mathbf{a}_{vu} - (\mathbf{n} \cdot \mathbf{a}_{vu})\mathbf{n} + \nabla_v((\mathbf{n}_u \cdot \mathbf{a})\mathbf{n}).$$

But since we are taking the tangential component, we can forget about differentiating the coefficient of **n**. Moreover, since **n** is a unit vector, $\mathbf{n}_v$ is already tangential, so we get:

$$\nabla_v \nabla_u \mathbf{a} = \mathbf{a}_{vu} - (\mathbf{n} \cdot \mathbf{a}_{vu})\mathbf{n} + (\mathbf{n}_u \cdot \mathbf{a})\mathbf{n}_v$$

Interchanging the roles of $u$ and $v$ and using the symmetry of the second derivative $\mathbf{a}_{uv} = \mathbf{a}_{vu}$ we get

$$\nabla_v \nabla_u \mathbf{a} - \nabla_u \nabla_v \mathbf{a} = (\mathbf{n}_u \cdot \mathbf{a})\mathbf{n}_v - (\mathbf{n}_v \cdot \mathbf{a})\mathbf{n}_u = (\mathbf{n}_u \wedge \mathbf{n}_v) \wedge \mathbf{a}.$$

Now

$$\mathbf{n}_u \wedge \mathbf{n}_v = \lambda \mathbf{n} \qquad (5)$$

so we see that $\nabla_v \nabla_u - \nabla_u \nabla_v$ acting on **a** rotates it in the tangent plane by 90° and multiplies by $\lambda$, where $\lambda$ is intrinsic. Now from (5),

$$\lambda \mathbf{n} \cdot \mathbf{r}_u \wedge \mathbf{r}_v = (\mathbf{n}_u \wedge \mathbf{n}_v) \cdot (\mathbf{r}_u \wedge \mathbf{r}_v) = (\mathbf{n}_u \cdot \mathbf{r}_u)(\mathbf{n}_v \cdot \mathbf{r}_v) - (\mathbf{n}_u \cdot \mathbf{r}_v)(\mathbf{n}_v \cdot \mathbf{r}_u) = LN - M^2$$

but also

$$\mathbf{n} \cdot \mathbf{r}_u \wedge \mathbf{r}_v = \sqrt{EG - F^2}$$

which gives

$$\lambda = (LN - M^2)/\sqrt{EG - F^2}. \qquad (6)$$

It follows that $LN - M^2$ and hence $K$ depends only on the first fundamental form.

## The Gauss-Bonnet theorem

One of the beautiful features of the Gaussian curvature is that it can be used to determine the topology of a closed orientable surface – more precisely we can determine the Euler characteristic by integrating $K$ over the surface. We shall do this by using a triangulation and summing the integrals over the triangles, but the boundary terms involve another intrinsic invariant of a curve in a surface:

**Definition 22** *The geodesic curvature* $\kappa_g$ *of a smooth curve in* $X$ *is defined by*

$$\kappa_g = \mathbf{t}' \cdot (\mathbf{n} \wedge \mathbf{t})$$

*where* $\mathbf{t}$ *is the unit tangent vector of the curve, which is parametrized by arc length.*

This is the tangential derivative of the unit tangent vector $\mathbf{t}$ and so is intrinsic.

The first version of Gauss-Bonnet is:

**Theorem 4.4** *Let* $\gamma$ *be a smooth simple closed curve on a coordinate neighbourhood of a surface* $X$ *enclosing a region* $R$*, then*

$$\int_\gamma \kappa_g ds = 2\pi - \int_R K dA$$

*where* $\kappa_g$ *is the geodesic curvature of* $\gamma$*,* $ds$ *is the element of arc-length of* $\gamma$*,* $K$ *is the Gaussian curvature of* $X$ *and* $dA$ *the element of area of* $X$*.*

**Proof:** Recall Stokes' theorem in $\mathbf{R}^3$:

$$\int_C \mathbf{a} \cdot d\mathbf{s} = \int_S \operatorname{curl} \mathbf{a} \cdot d\mathbf{S}$$

for a curve $C$ spanning a surface $S$. In the $xy$ plane with $\mathbf{a} = (P, Q, 0)$ this becomes Green's formula

$$\int_\gamma (Pu' + Qv')dt = \int_R (Q_u - P_v) du dv \tag{7}$$

Now choose a unit length tangent vector field, for example $\mathbf{e} = \mathbf{r}_u/\sqrt{E}$. Then $\mathbf{e}, \mathbf{n} \wedge \mathbf{e}$ is an orthonormal basis for each tangent space. Since $\mathbf{e}$ has unit length, $\nabla_u \mathbf{e}$ is tangential and orthogonal to $\mathbf{e}$ so there are functions $P, Q$ such that

$$\nabla_u \mathbf{e} = P\, \mathbf{n} \wedge \mathbf{e}, \qquad \nabla_v \mathbf{e} = Q\, \mathbf{n} \wedge \mathbf{e}.$$

In Green's formula, take $\mathbf{a} = (P, Q, 0)$ then the left hand side of (7) is

$$\int_\gamma (u' \nabla_u \mathbf{e} + v' \nabla_v \mathbf{e}) \cdot (\mathbf{n} \wedge \mathbf{e}) ds = \int_\gamma \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e}) ds \tag{8}$$

Let $\mathbf{t}$ be the unit tangent to $\gamma$, and write it relative to the orthonormal basis

$$\mathbf{t} = \cos\theta\, \mathbf{e} + \sin\theta\, \mathbf{n} \wedge \mathbf{e}.$$

So

$$\mathbf{t}' \cdot (\mathbf{n} \wedge \mathbf{e}) = \cos\theta\, \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e}) + \cos\theta\, \theta'.$$

The geodesic curvature of $\gamma$ is defined by $\kappa_g = \mathbf{t}' \cdot (\mathbf{n} \wedge \mathbf{t})$ so

$$\mathbf{t}' = \alpha \mathbf{n} + \kappa_g \mathbf{n} \wedge \mathbf{t} = \alpha \mathbf{n} + \kappa_g (\cos\theta\, \mathbf{n} \wedge \mathbf{e} - \sin\theta\, \mathbf{e})$$

and so

$$\kappa_g = \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e}) + \theta'.$$

We can therefore write (8) as

$$\int_\gamma (\kappa_g - \theta') ds$$

and as $\theta$ changes by $2\pi$ on going round the curve, this is

$$\int_\gamma \kappa_g ds - 2\pi.$$

To compute the right hand side of (7), note that

$$\nabla_v \nabla_u \mathbf{e} = \nabla_v (P\, \mathbf{n} \wedge \mathbf{e}) = P_v\, \mathbf{n} \wedge \mathbf{e} + P\, \mathbf{n} \wedge \nabla_v \mathbf{e} = P_v\, \mathbf{n} \wedge \mathbf{e} + PQ\, \mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{e})$$

since $\mathbf{n}_v \wedge \mathbf{e}$ is normal. Interchanging the roles of $u$ and $v$ and subtracting we obtain

$$(\nabla_v \nabla_u - \nabla_u \nabla_v)\mathbf{e} = (P_v - Q_u)\mathbf{n} \wedge \mathbf{e}$$

and from (6) this is equal to $K\sqrt{EG - F^2}$.

Applying Green's theorem and using $dA = \sqrt{EG - F^2} du dv$ gives the result. □

> Note that the extrinsic normal was only used to define $\mathbf{n} \wedge \mathbf{e}$ which is one of the two unit tangent vectors to $X$ orthogonal to $\mathbf{e}$. If the surface is orientable we can systematically make a choice and then the proof is intrinsic.

If the curve $\gamma$ is piecewise smooth – a curvilinear polygon – then $\theta$ jumps by the external angle $\delta_i$ at each vertex, so the integral of $\theta'$ which is $2\pi$ in the theorem is replaced by

$$\int_\gamma \theta' ds = 2\pi - \sum_i \delta_i = \sum_i \alpha_i - (n-2)\pi$$

where $\alpha_i$ are the internal angles. The Gauss-Bonnet theorem gives in particular:

**Theorem 4.5** *The sum of the angles of a curvilinear triangle is*

$$\pi + \int_R K dA + \int_\gamma \kappa_g ds.$$

**Examples:**

1. In the plane, a line has constant unit tangent vector and so $\kappa_g = 0$. Since the Gaussian curvature is zero too this says that the sum of the angles of a triangle is $\pi$.

2. A great circle on the unit sphere also has $\kappa_g$ zero, for example if $\gamma(s) = (\cos s, \sin s, 0)$, then $\mathbf{t} = (-\sin s, \cos s, 0)$ and $\mathbf{t}' = -(\cos s, \sin s, 0)$ which is normal to the sphere. Since here $K = 1$, we have, for the triangle $\Delta$ with angles $A, B, C$

$$\alpha + \beta + \gamma = \pi + \text{Area}(ABC).$$

Here is the most interesting version of Gauss-Bonnet:

**Theorem 4.6** *If $X$ is a smooth orientable closed surface with a Riemannian metric, then*

$$\int_X K dA = 2\pi\chi(X)$$

**Proof:** Take a smooth triangulation so that each triangle is inside a coordinate neighbourhood and apply Theorem 4.5 and add. The integrals of $\kappa_g$ on the edges cancel because the orientation on the edge from adjacent triangles is opposite (this is for Green's theorem – we use the anticlockwise orientation on $\gamma$). The theorem gives the total sum of internal angles as

$$\pi F + \int_X K dA.$$

But around each vertex the internal angles add to $2\pi$ so we have

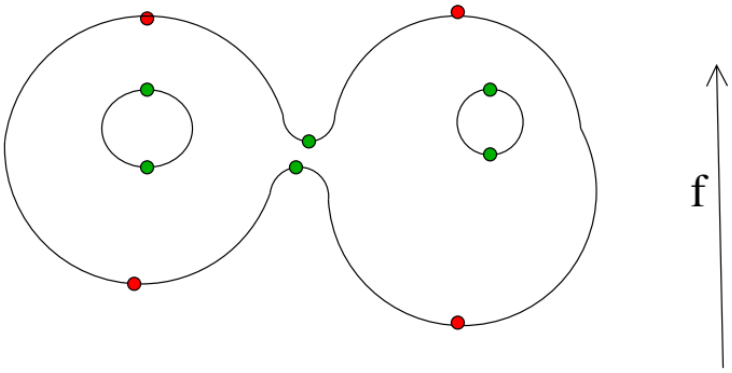
$$2\pi V = \pi F + \int_X K dA$$

and as our faces are triangles whose sides meet in pairs there are $3F/2$ edges. Hence

$$2\pi\chi(X) = 2\pi(V - E + F) = \pi F + \int_X K dA - 3\pi F + 2\pi F = \int_X K dA.$$

$\square$

The Gauss-Bonnet theorem and its method of proof give another formula for the Euler characteristic, involving smooth real-valued functions $f : X \to \mathbf{R}$ on a closed surface $X$. Since $X$ is compact, $f$ certainly has a maximum and a minimum, but may have other critical points too. Think of a surface in $\mathbf{R}^3$ and the function $f$ given by its height above a plane:



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This has 2 maxima, 2 minima and 6 saddle points. We shall be able to calculate the Euler characteristic from these numbers.

First recall that a smooth function $f(u,v)$ has a critical point at $a$ if

$$f_u(a) = f_v(a) = 0.$$

Because of the chain rule, this condition is independent of coordinates: if $u = u(x,y), v = v(x,y)$ then

$$f_x = f_u u_x + f_v v_x, \qquad f_y = f_u u_y + f_v v_y$$

so $f_u$ and $f_v$ vanish if and only if $f_x$ and $f_y$ vanish. This means we can unambiguously talk about the critical points of a smooth function on a surface $X$.

The Hessian matrix

$$\begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

*at a critical point* transforms like

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

and so

$$(f_{uu} f_{vv} - f_{uv}^2) = (u_x v_y - u_y v_x)^2 (f_{uu} f_{vv} - f_{uv}^2)$$

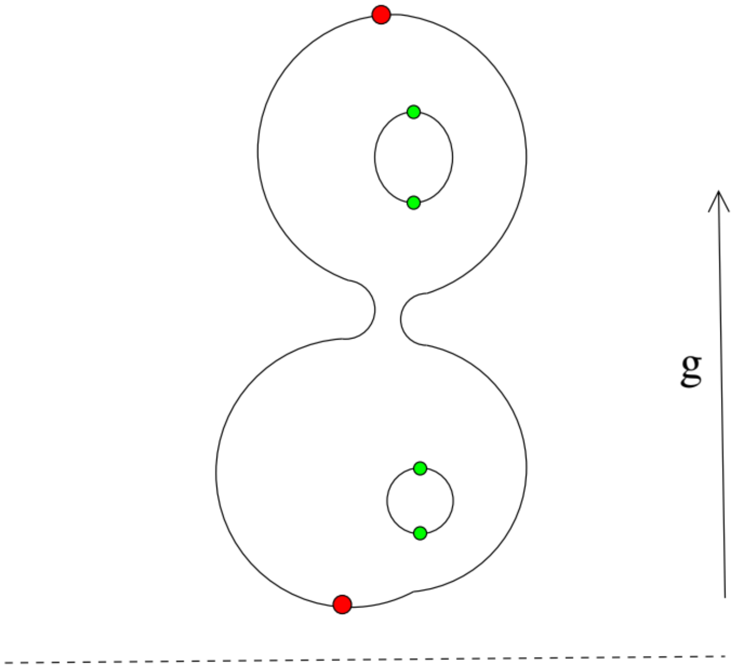
therefore to say that the determinant of the Hessian is non-zero, or positive or negative, is again independent of the choice of coordinate.

**Definition 23** *A function $f$ on a surface $X$ has a nondegenerate critical point at $a \in X$ if its Hessian at $a$ is invertible.*

We know from calculus that if $f_{uu} f_{vv} - f_{uv}^2 > 0$ and $f_{uu} > 0$ we have a local minimum, if $f_{uu} < 0$ a local maximum and if $f_{uu} f_{vv} - f_{uv}^2 < 0$ a saddle point. The theorem is the following:

**Theorem 4.7** *Let $f$ be a smooth function on a closed surface $X$ with nondegenerate critical points, then the Euler characteristic $\chi(X)$ is the number of local maxima and minima minus the number of saddle points.*

In the picture, we have $\chi(X) = 4 - 6 = -2$ which is correct for the connected sum of two tori. If we turn it on its side we get one maximum, one minimum and 4 saddle points again giving the same value: $2 - 4 = -2$.



**Proof:** Given a function $f$ on $X$ we can define its gradient vector field:

$$\mathbf{a} = \frac{1}{EG - F^2}[(Gf_u - Ff_v)\mathbf{r}_u + (Ef_v - Ff_u)\mathbf{r}_v]$$

which is normal to the contour lines of $f$. Away from the critical points we can normalize it to get a unit vector field $\mathbf{e}$. Surround each critical point by a small closed curve $\gamma_i$ enclosing a disc $R_i$. Let $Y$ be the complement of the discs, then from the argument of Theorem 4.4

$$\int_Y K dA = -\sum_i \int_{\gamma_i} \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e}) ds$$

using the negative sign because $Y$ is outside $R_i$.

Inside $R_i$ we choose a unit vector field $\mathbf{f}$ and then we get

$$\int_{R_i} K dA = \int_{\gamma_i} \mathbf{f}' \cdot (\mathbf{n} \wedge \mathbf{f}) ds$$

so adding gives

$$\int_X K dA = \sum_i \int_{\gamma_i} [\mathbf{f}' \cdot (\mathbf{n} \wedge \mathbf{f}) - \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e})] ds.$$

From the proof of the theorem we had

$$\kappa_g = \mathbf{e}' \cdot (\mathbf{n} \wedge \mathbf{e}) + \theta' = \mathbf{f}' \cdot (\mathbf{n} \wedge \mathbf{f}) + \phi'$$

where $\theta$ is the angle between $\gamma'$ and $\mathbf{e}$ and $\phi$ between $\gamma'$ and $\mathbf{f}$. So the contribution is just the change in angle between the vector field $\mathbf{e}$ and a fixed one $\mathbf{f}$ which extends. This is an integer multiple of $2\pi$ so we can evaluate it by deforming to the standard Euclidean case. A local minimum is $f = x^2 + y^2$ which gives

$$\mathbf{e} = (\cos\theta, \sin\theta)$$

and contributes $+1$, as does the local minimum $-(\cos\theta, \sin\theta)$. For a saddle point $f = x^2 - y^2$ which gives

$$\mathbf{e} = (\cos\theta, -\sin\theta) = (\cos(-\theta), \sin(-\theta))$$

and contributes $-1$. □

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