

Geodesics

Geodesics on a surface are curves which are the analogues of straight lines in the plane. Lines can be thought of in two ways:

- shortest curves
- straightest curves

The first point of view says that a straight line minimizes the distance between any two of its points. Conceptually this leads to the idea of stretching a string between two points on a surface until it tightens, and this certainly is one approach to geodesics. The second approach is however generally easier. A line is straightest because its tangent vector doesn't change – it is constant along the line. We generalize this to a curve on a surface by insisting that the component of \mathbf{t}' tangential to the surface should vanish. Or...

Definition 24 A *geodesic* on a surface X is a curve $\gamma(s)$ on X such that \mathbf{t}' is normal to the surface.

From Definition 22 this is the same as saying that the geodesic curvature vanishes.

The general problem of finding geodesics on a surface is very complicated. The case of the ellipsoid is a famous example, needing hyperelliptic functions to solve it – integrals of $dz/\sqrt{p(z)}$ where $p(z)$ is a polynomial of degree 6. But there are cheap ways to find some of them, as in these examples:

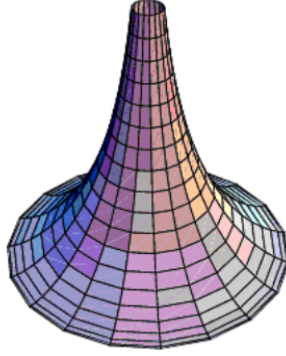
Examples:

- 1) The normal to a curve in the plane is parallel to the plane, so the condition that \mathbf{t}' is normal to the plane means $\mathbf{t}' = 0$ which integrates to $\mathbf{r} = s\mathbf{a} + \mathbf{c}$, the equation of a straight line. Geodesics in the plane really are straight lines, then.
- 2) Take the unit sphere and a plane section through the origin. We saw earlier that $\kappa_g = 0$ here.
- 3) Similarly, any plane of symmetry intersects a surface in a geodesic, because the normal to the surface at such a point must be invariant under reflection in the plane of symmetry and hence lie in that plane. It is orthogonal to the tangent vector of the curve of intersection and so \mathbf{t}' points normally.

A useful class of examples is provided by a surface of revolution

$$\mathbf{r}(u, v) = f(u)(\cos v \mathbf{i} + \sin v \mathbf{j}) + u\mathbf{k}$$

The reflection $(x, y, z) \mapsto (x, -y, z)$ maps the surface to itself, as, by symmetry, does any reflection in a plane containing the z -axis. So the *meridians* $v = \text{const.}$ are geodesics:



To find the geodesics in general we need to solve a nonlinear system of ordinary differential equations:

Proposition 4.8 *A curve $\gamma(s) = (u(s), v(s))$ on a surface parametrized by arc length is a geodesic if and only if*

$$\begin{aligned}\frac{d}{ds}(Eu' + Fv') &= \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2) \\ \frac{d}{ds}(Fu' + Gv') &= \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2)\end{aligned}$$

Proof: We have for the curve γ

$$\mathbf{t} = \mathbf{r}_u u' + \mathbf{r}_v v'$$

and it is a geodesic if and only if \mathbf{t}' is normal i.e.

$$\mathbf{t}' \cdot \mathbf{r}_u = \mathbf{t}' \cdot \mathbf{r}_v = 0.$$

Now

$$\mathbf{t}' \cdot \mathbf{r}_u = (\mathbf{t} \cdot \mathbf{r}_u)' - \mathbf{t} \cdot \mathbf{r}'_u$$

so the first equation is

$$(\mathbf{t} \cdot \mathbf{r}_u)' = \mathbf{t} \cdot \mathbf{r}'_u.$$

The left hand side is

$$\frac{d}{ds}((\mathbf{r}_u u' + \mathbf{r}_v v') \cdot \mathbf{r}_u) = \frac{d}{ds}(Eu' + Fv')$$

an the right hand side is

$$\begin{aligned}\mathbf{t} \cdot (\mathbf{r}_{uu} u' + \mathbf{r}_{uv} v') &= \mathbf{r}_u \cdot \mathbf{r}_{uu} u'^2 + (\mathbf{r}_v \cdot \mathbf{r}_{uu} + \mathbf{r}_u \cdot \mathbf{r}_{uv}) u'v' + \mathbf{r}_v \cdot \mathbf{r}_{uv} v'^2 \\ &= \frac{1}{2} E_u u'^2 + (\mathbf{r}_v \cdot \mathbf{r}_u)_u u'v' + \frac{1}{2} G_u v'^2 \\ &= \frac{1}{2} (E_u u'^2 + 2F_u u'v' + G_u v'^2)\end{aligned}$$

The other equation follows similarly. □

It is clear from 4.8 that geodesics only depend on the first fundamental form, so that geodesics can be defined for abstract surfaces and moreover an isometry takes geodesics to geodesics.

Examples:

1) The plane: $E = 1, F = 0, G = 1$ in Cartesian coordinates, so the geodesic equations are

$$x'' = 0 = y''$$

which gives straight lines

$$x = \alpha_1 s + \beta_1, \quad y = \alpha_2 s + \beta_2.$$

2) The cylinder

$$\mathbf{r}(u, v) = a(\cos v \mathbf{i} + \sin v \mathbf{j}) + u\mathbf{k}$$

has first fundamental form

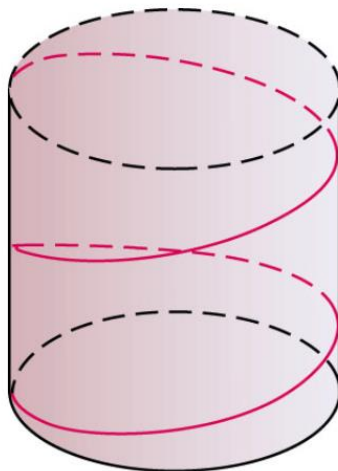
$$du^2 + a^2 dv^2 = du^2 + d(av)^2.$$

This is isometric to the plane so the geodesics are of the form

$$u = \alpha_1 s + \beta_1, \quad v = \alpha_2 s + \beta_2$$

which gives a helix

$$\boldsymbol{\gamma} = a(\cos(\alpha_2 s + \beta_2) \mathbf{i} + \sin(\alpha_2 s + \beta_2) \mathbf{j}) + (\alpha_1 s + \beta_1) \mathbf{k}$$



The differential equation for geodesics gives us the following general fact:

Proposition 4.9 *Through each point P on a surface and in each direction at P there passes a unique geodesic.*

Proof: We are solving a differential equation of the form

$$u'' = a(u, v, u', v'), \quad v'' = b(u, v, u', v')$$

or equivalently a first order system

$$\begin{aligned}
u' &= p \\
v' &= q \\
p' &= a(u, v, p, q) \\
q' &= b(u, v, p, q)
\end{aligned}$$

and the Cauchy existence theorem (see Appendix B) gives a unique solution with initial conditions (u, v, p, q) , namely the point of origin and the direction. \square

Example: Given a point \mathbf{a} on the unit sphere and a tangential direction \mathbf{b} the span of \mathbf{a}, \mathbf{b} is a plane through the origin which meets the sphere in a great circle through \mathbf{a} with tangent \mathbf{b} . Thus *every* geodesic is a great circle.

There is one case – a surface of revolution – where the geodesic equations can be “solved”, or anyway, reduced to a single integration. We have

$$E = 1 + f'(u)^2, \quad F = 0, \quad G = f(u)^2$$

and the equations become

$$\begin{aligned}
\frac{d}{ds}((1 + f'^2)u') &= f'(f''u'^2 + fv'^2) \\
\frac{d}{ds}(f^2v') &= 0
\end{aligned}$$

We ignore the first equation – it is equivalent to a more obvious fact below. The second says that

$$f^2v' = c \tag{9}$$

where c is a constant. Now use the fact that the curve is parametrized by arc length (this is an “integral” of the equations), and we get

$$(1 + f'^2)u'^2 + f^2v'^2 = 1 \tag{10}$$

Substitute for v' from (9) in (10) to get

$$(1 + f'^2)u'^2 + \frac{c^2}{f^2} = 1$$

and then

$$s = \int f \sqrt{\frac{1 + f'^2}{f^2 - c^2}} du$$

which is “only” an integration. Having solved this by $u = h(s)$, v can be determined by a further integration from (9):

$$v(s) = \int \frac{c}{f(h(t))^2} dt.$$

If we are only interested in the curve and not its arclength parametrization, then (9) and (10) give

$$(1 + f'(u)^2) \left(\frac{du}{dv} \right)^2 + f(u)^2 = \frac{f(u)^4}{c^2}$$

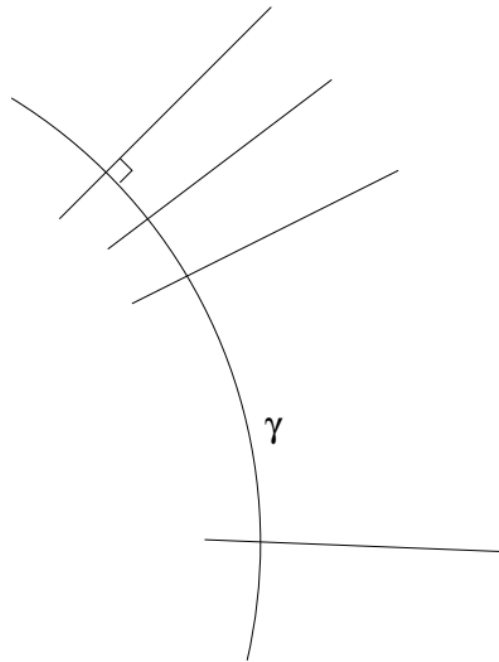
which reduces to the single integration

$$v = \int \frac{c}{f(u)} \sqrt{\frac{1 + f'(u)^2}{f(u)^2 - c^2}} du.$$

Gaussian curvature revisited

We may not be able to solve the geodesic equations explicitly, but existence of geodesics through a given point and in a given direction give rise to various natural coordinate systems, modelled on Cartesian coordinates. Here is one: choose a geodesic γ parametrized by arc length. Through the point $\gamma(v)$ take the geodesic $\gamma_v(s)$ which intersects γ orthogonally, and define

$$\mathbf{r}(u, v) = \gamma_v(u).$$



Since \mathbf{r}_u and \mathbf{r}_v are orthogonal at $u = 0$ they are linearly independent in a neighbourhood and so are good coordinates.

Now the curves $v = \text{const.}$ are parametrized by arc length, so $E = 1$. These curves are also geodesics and u is arc length so in the second geodesic equation

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2)$$

we put $v = \text{const.}$ and $u = s$ which, with $E = 1$, gives $F_u = 0$. But F vanishes at $u = 0$ because the two geodesics are orthogonal there, hence $F = 0$ and the first fundamental form is

$$du^2 + G(u, v)dv^2.$$

In this form the Gaussian curvature is simple:

Proposition 4.10 *The Gaussian curvature of the metric $du^2 + G(u, v)dv^2$ is*

$$K = -G^{-1/2}(G^{1/2})_{uu}$$

Examples:

1. For the plane $dx^2 + dy^2$, $G = 1$ and $K = 0$.
2. For the unit sphere with first fundamental form $du^2 + \sin^2 u dv^2$, $G = \sin^2 u$ so

$$K = -\frac{1}{\sin u}(\sin u)_{uu} = \frac{1}{\sin u} \sin u = 1.$$

3. For the upper half-space with metric $(dx^2 + dy^2)/y^2$ put $u = \log y$ and $v = x$ and then we have $du^2 + e^{-2u}dv^2$, so that

$$K = -e^u(e^{-u})_{uu} = -e^u e^{-u} = -1.$$

Proof: Recall the tangential derivative ∇ : the tangential component of the ordinary derivative. Then since by construction \mathbf{r}_u is the unit tangent vector of a geodesic, by the definition of a geodesic its u -derivative is normal so $\nabla_u \mathbf{r}_u = 0$.

Consider now $\nabla_v \mathbf{r}_u = A\mathbf{r}_u + B\mathbf{r}_v$. The dot product with \mathbf{r}_u gives

$$E_v/2 = \mathbf{r}_{vu} \cdot \mathbf{r}_u = A$$

but $E = 1$ so $A = 0$.

Using $E = 1$ and $F = 0$ the product with \mathbf{r}_v gives

$$G_u/2 = \mathbf{r}_v \cdot \mathbf{r}_{vu} = BG.$$

Now from (6)

$$(\nabla_v \nabla_u - \nabla_u \nabla_v)\mathbf{r}_u = K\sqrt{EG - F^2}\mathbf{n} \wedge \mathbf{r}_u = KG^{1/2}(\mathbf{r}_v G^{-1/2}) = K\mathbf{r}_v$$

But the left hand side (using $\nabla_u \mathbf{r}_v = \nabla_v \mathbf{r}_u$ which follows from $\mathbf{r}_{uv} = \mathbf{r}_{vu}$) is

$$-\nabla_u(G_u/2G)\mathbf{r}_v = -((G_u/2G)_u + (G_u/2G)^2)\mathbf{r}_v$$

which gives the result. \square

With this coordinate system we can characterize surfaces with *constant* Gaussian curvature:

Theorem 4.11 *A surface with $K = 0$ is locally isometric to the plane, with $K = 1$ locally isometric to the unit sphere and with $K = -1$ locally isometric to the upper half space with metric $(dx^2 + dy^2)/y^2$.*

Proof: Use the form $du^2 + Gdv^2$.

i) If $K = 0$ then $(G^{1/2})_{uu} = 0$ so $G^{1/2} = A(v)u + B(v)$. But at $u = 0$, \mathbf{r}_u and \mathbf{r}_v are unit so $B(v) = 1$. Also, the curve $u = 0$ is a geodesic – the initial curve γ – with v arc length. So the geodesic equation

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)$$

gives $0 = G_u(0, v)/2$ and this means in our case $A(v) = 0$. The first fundamental form is therefore $du^2 + dv^2$ and by 4.1 this is isometric to the plane.

ii) If $K = 1$, the equation for $G^{1/2}$ is

$$(G^{1/2})_{uu} + G^{1/2} = 0$$

which is solved by $G^{1/2} = A(v) \sin u + B(v) \cos u$. The boundary conditions give $G = \cos^2 u$ and the metric $du^2 + \cos^2 u dv^2$ – the sphere.

iii) If $K = -1$ we have $du^2 + \cosh^2 u dv^2$. The substitution $x = e^v \tanh u$, $y = e^v \operatorname{sech} u$ takes this into $(dx^2 + dy^2)/y^2$. \square

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