The hyperbolic plane

Isometries

We just saw that a metric of constant negative curvature is modelled on the upper half space H with metric

$$\frac{dx^2 + dy^2}{y^2}$$

which is called the *hyperbolic plane*. This is an abstract surface in the sense that we are not considering a first fundamental form coming from an embedding in $\mathbf{R}^3$, and yet it is concrete enough to be able to write down and see everything explicitly. First we consider the isometries from H to itself.

If $a, b, c, d \in \mathbf{R}$ and $ad - bc > 0$ then the Möbius transformation

$$z \mapsto w = \frac{az + b}{cz + d} \tag{11}$$

restricts to a smooth bijection from H to H with smooth inverse

$$w \mapsto z = \frac{dw - b}{-cw + a}.$$

If we substitute

$$w = \frac{az + b}{cz + d} \text{ and } dw = \left(\frac{a}{cz + d} - \frac{c(az + b)}{(cz + d)^2} \right) dz = \frac{(ad - bc)}{(cz + d)^2} dz$$

into

$$\frac{du^2 + dv^2}{v^2} = \frac{4|dw|^2}{|w - \bar{w}|^2}$$

we get

$$\frac{4(ad - bc)^2 |dz|^2}{|(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)|^2} = \frac{4(ad - bc)^2 |dz|^2}{|(ad - bc)(z - \bar{z})|^2} = \frac{4|dz|^2}{|z - \bar{z}|^2} = \frac{dx^2 + dy^2}{y^2}.$$

Thus this Möbius transformation is an isometry from H to H. So is the transformation $z \mapsto -\bar{z}$, and hence the composition

$$z \mapsto \frac{b - a\bar{z}}{d - c\bar{z}} \tag{12}$$

is also an isometry from H to H. In fact (11) and (12) give all the isometries of H, as we shall see later.

we saw that the unit disc D with the metric

$$\frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}$$

is isometric to H, so any statements about H transfer also to D. Sometimes the picture is easier in one model or the other. The isometries $f : D \to D$ of the unit disc model of the hyperbolic plane are also Möbius transformations, if they preserve orientations, or compositions of Möbius transformations with $z \mapsto \bar{z}$ if they reverse orientations. The Möbius transformations which map D to itself are those of the form

$$z \mapsto w = e^{i\theta}\left(\frac{z - a}{1 - \bar{a}z}\right)$$

where $a \in D$ and $\theta \in \mathbf{R}$. They are isometries because substituting for w and

$$dw = e^{i\theta}\frac{(1 - |a|^2)}{(1 - \bar{a}z)^2}dz$$

in $4(1 - |w|^2)^{-2}|dw|^2$ gives $4(1 - |z|^2)^{-2}|dz|^2$.

Notice that the group $\mathrm{Isom}(H)$ of isometries of H acts transitively on H because if $a + ib \in H$ then $b > 0$ so the transformation

$$z \mapsto bz + a$$

is an isometry of H which takes i to $a + ib$. Similarly the group $\mathrm{Isom}(D)$ of isometries of D acts transitively on D since if $a \in D$ then the isometry

$$z \mapsto \frac{z - a}{1 - \bar{a}z}$$

maps a to 0. Notice also that the subgroup of $\mathrm{Isom}(D)$ consisting of those isometries which fix 0 contains all the rotations

$$z \mapsto e^{i\theta}z$$

about 0 as well as $z \mapsto \bar{z}$.

Geodesics

The hyperbolic plane is a case where the geodesic equations can be easily solved: since $E = G = 1/y^2$ and $F = 0$, and these are independent of x, the first geodesic equation

$$\frac{d}{ds}(Eu' + Fv') = \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2)$$

becomes

$$\frac{d}{ds}(\frac{x'}{y^2}) = 0$$

and so

$$x' = cy^2. \tag{13}$$

We also know that parametrization is by arc length in these equations so

$$\frac{x'^2 + y'^2}{y^2} = 1 \tag{14}$$

If $c = 0$ we get $x = const.$, which is a vertical line. Suppose $c \neq 0$, then from (13) and (14) we have

$$\frac{dy}{dx} = \sqrt{\frac{y^2 - c^2y^4}{c^2y^4}}$$

or

$$\frac{cydy}{\sqrt{1 - c^2y^2}} = dx$$

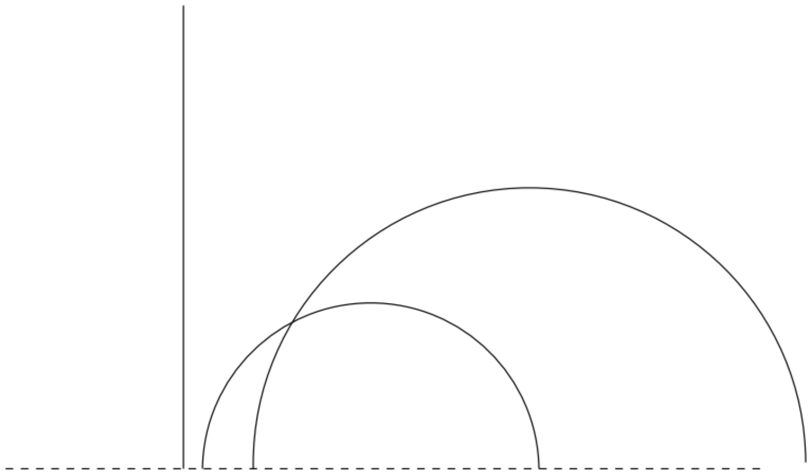
which integrates directly to

$$-c^{-1}\sqrt{1 - c^2y^2} = x - a$$

or

$$(x - a)^2 + y^2 = 1/c^2$$

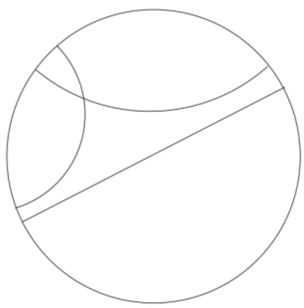
which is a semicircle centred on the real axis.



The isometry from H to D given by

$$w \mapsto z = \frac{w - i}{w + i}$$

takes geodesics to geodesics (since it is an isometry) and it is the restriction to H of a Möbius transformation $\mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ which takes circles and lines to circles and lines, preserves angles and maps the real axis to the unit circle in $\mathbf{C}$. It therefore follows that the geodesics in D are the circles and lines in D which meet the unit circle at right angles.



Using geodesics we can now show that any isometry is a Möbius transformation as above. So suppose that $F : D \to D$ is an isometry. Take a Möbius isometry G taking $F(0)$ to 0, then we need to prove that GF is Möbius. This is an isometry fixing 0, so it takes geodesics through 0 to geodesics through 0. It preserves angles, so acts on those geodesics by a rotation or reflection. It also preserves distance so it takes a point on a geodesic a distance r from the origin to another point at the same distance. However, as we noted above, each rotation $R : z \mapsto e^{i\theta} z$ is a Möbius isometry, so composing with this we see that $RGF = 1$ and $F = (RG)^{-1}$ is a Möbius isometry.

Angles and distances

Hyperbolic angles in H and in D are the same as Euclidean angles, since their first fundamental forms satisfy $E = G$ and $F = 0$. Distances between points are given by the lengths of geodesics joining the points. Since the interval $(-1, 1)$ is a geodesic in the unit disc D, the distance from 0 to any $x \in (0, 1)$ is given by the hyperbolic length of the line segment $[0, x]$, which is

$$\int_0^x \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt = \int_0^x \frac{dt}{1-t^2} = 2\tanh^{-1}x$$

where $u(t) = t$ and $v(t) = 0$ and $E = G = (1 - u^2 - v^2)^{-2}$ and $F = 0$. Given any $a, b \in D$ we can choose $\theta \in \mathbf{R}$ such that

$$e^{i\theta}\frac{b-a}{1-\bar{a}b} = \left|\frac{b-a}{1-\bar{a}b}\right|$$

is real and positive, so its distance from 0 is

$$2\tanh^{-1}\left|\frac{b-a}{1-\bar{a}b}\right|.$$

Since the isometry

$$z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

preserves distances and takes a to 0 and b to $e^{i\theta}(b-a)/(1-\bar{a}b)$, it follows that the hyperbolic distance from a to b in D is

$$d_D(a, b) = 2\tanh^{-1}\left|\frac{b-a}{1-\bar{a}b}\right|.$$

We can work out hyperbolic distances in H in a similar way by first calculating the distance from i to λi for $\lambda \in [1, \infty)$ as the length of the geodesic from i to λi given by the imaginary axis, which is

$$\int_1^\lambda \frac{dt}{t} = \log\lambda,$$

and then given $a, b \in H$ finding an isometry of H which takes a to i and b to λi for some $\lambda \in [1, \infty)$. Alternatively, since we have an isometry from H to D given by

$$w \mapsto z = \frac{w-i}{w+i},$$

the hyperbolic distance between points $a, b \in H$ is equal to the hyperbolic distance between the corresponding points $(a-i)/(a+i)$ and $(b-i)/(b+i)$ in D, which is

$$d_H(a,b) = d_D(\frac{a-i}{a+i}, \frac{b-i}{b+i}) = 2\tanh^{-1}\left|\frac{(b-i)(a+i)-(a-i)(b+i)}{\overline{(a+i)}(b+i)-\overline{(a-i)}(b-i)}\right| = 2\tanh^{-1}\left|\frac{b-a}{b-\bar{a}}\right|.$$

Hyperbolic Geometry: an introduction

Refresher about Möbius maps

Möbius maps for hyperbolic geometry are as important as rotations and translations are in Euclidean geometry. Indeed, for the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and the hyperbolic disc $D = \{z \in \mathbb{C} : |z| < 1\}$, a subgroup of the Möbius maps will turn out to be the group of all isometries.

$$D \to \mathbb{H},\ z \mapsto \tau(z) = \frac{iz+i}{-z+1} \qquad \mathbb{H} \to D,\ z \mapsto \tau^{-1}(z) = \frac{z-i}{z+i}.$$

Corollary *The biholomorphisms $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1$ are precisely the Möbius maps*

$$\varphi(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (we often rescale numerator and denominator so that $ad - bc = 1$). In particular, $\varphi(\infty) = a/c$, $\varphi(-d/c) = \infty$. These form a group isomorphic to

$$PSL(2,\mathbb{C}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{C}, ad - bc = 1\}/\pm\text{Identity}.$$

We recall some useful properties about Möbius maps:

Lemma

(1) *Möbius maps preserve angles*
(2) *Möbius maps are generated by translations $\varphi(z) = z + b$, dilations $\varphi(z) = az$ (for $a \neq 0$), and inversions $\varphi(z) = 1/z$ (which corresponds to inversion in the unit circle followed by reflection in the real axis).*
(3) *Möbius maps send circles to circles (where we allow straight lines, thought of as circles of infinite radius).*
(4) *Given any three distinct points $z_0, z_1, z_2 \in \mathbb{C}P^1$, there is a Möbius map φ such that $\varphi(z_0) = 0$, $\varphi(z_1) = 1$, $\varphi(z_2) = \infty$.*
(5) *A Möbius map is uniquely determined by where it sends three points, for example it is determined by the values $\varphi(0)$, $\varphi(1)$, $\varphi(\infty)$.*

---

[1]Second refresher on hyperbolic functions: $\tanh t = \frac{\sinh t}{\cosh t}$ and $\operatorname{sech} t = \frac{1}{\cosh t}$ have derivatives $\tanh'(t) = \operatorname{sech}^2(t)$ and $\operatorname{sech}'(t) = -\frac{\sinh t}{\cosh^2 t} = -\tanh t \operatorname{sech} t$. We will use the useful identity

$$\tanh^2 y + \operatorname{sech}^2 y = 1$$

(which follows from $\cosh^2 y - \sinh^2 y = 1$). E.g. this shows we can invert the above change of variables: $\tilde{x}^2 + \tilde{y}^2 = e^{2x}$ so we recover x, then we recover y. Motivation for trying that change of variables is because of: $\cosh^2 y\, dx^2 + dy^2 = \cosh^2 y\,(dx^2 + \operatorname{sech}^2 y\, dy^2) = \cosh^2 y\,(dx^2 + \frac{d(\tanh y)^2}{\operatorname{sech}^2 y}) = \frac{1}{\operatorname{sech}^4 y}(\operatorname{sech}^2 y\, dx^2 + d(\tanh y)^2)$.

(6) *The Möbius maps with $\varphi(\mathbb{H}) = \mathbb{H}$, with $ad - bc = 1$, are those with a, b, c, d all real:*

$$\boxed{\text{Möb}(\mathbb{H}) = \left\{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc = 1\right\} \cong \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{R}, \det = 1\}/\pm Id = PSL(2,\mathbb{R})}$$

(7) *Given $z \in \mathbb{H}$ there is a Möbius map φ with $\varphi(\mathbb{H}) = \mathbb{H}$ and $\varphi(i) = z$.*

Proof. For (1): Möbius maps φ are holomorphic, so the derivative $D_z\varphi$ is a composition of a scaling and a rotation (see Analysis handout), so it preserves angles.

For (2), when $c \neq 0$,

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c(cz+d)}$$

so it is a composition of translations, dilations, inversions. The case $c = 0$ is even easier. Conversely, translations, dilations and inversions are Möbius maps.

For (3): it is enough by (2) to check that translations, dilations and inversions send circles to circles, and an easy check shows that they do.

For (4), we can even write an explicit formula:

$$\varphi(z) = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)}$$

For (5): suppose a Möbius map ψ sends distinct points w_0, w_1, w_2 to z_0, z_1, z_2. Let φ be a map as in (4). Then $\varphi \circ \psi$ is a Möbius map which sends w_0, w_1, w_2 to $0, 1, \infty$. Let α be the inverse of a map as in (4) for the points w_0, w_1, w_2, so α sends $0, 1, \infty$ to w_0, w_1, w_2. Then $\varphi \circ \psi \circ \alpha$ is a Möbius map which fixes $0, 1, \infty$. By looking at the equations this implies for the constants a, b, c, d which define the map, you easily deduce that: $b = 0$ (fixes 0), $c = 0$ (fixes ∞) and so $a/d = 1$ (fixes 1), so this map is the identity. So $\psi = \varphi^{-1}\alpha^{-1}$ is determined.

For (6): since Möbius maps are biholomorphisms, if $\varphi(\mathbb{H}) = \mathbb{H}$ then the restriction $\varphi : \mathbb{H} \to \mathbb{H}$ is a biholomorphism as well. By continuity, the boundary $\mathbb{R}$ of $\mathbb{H}$ has to be mapped into itself. As $\varphi(0) = r_0$, $\varphi(1) = r_1$, $\varphi(\infty) = r_2$ are real numbers, by the above formula we have $\varphi^{-1}(z) = \frac{(z-r_0)(r_1-r_2)}{(z-r_2)(r_1-r_0)}$. This is in $PSL(2,\mathbb{R})$ so its inverse φ is also in $PSL(2,\mathbb{R})$. Conversely, if a, b, c, d are real, then $\varphi(\mathbb{R}) = \mathbb{R}$, hence φ permutes the two connected components of $\mathbb{C} \setminus \mathbb{R}$ (since φ is a biholomorphism). So $\varphi(\mathbb{H}) = \mathbb{H}$ precisely if $\varphi(i) \in \mathbb{H}$. So we check the sign: $\operatorname{sign} \operatorname{Im} \varphi(i) = \operatorname{sign}(ad - bc)$. So $\varphi(\mathbb{H}) = \mathbb{H}$ precisely if $\det > 0$.

For (7): $z = b + ia$ in terms of real and imaginary parts $b, a \in \mathbb{R}$, then $\varphi(z) = az + b$ works (so take $c = 0$ and $d = 1$). □

Exercise. *Recall the hyperbolic disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is isometric to $\mathbb{H}$ (using the hyperbolic metric $\frac{4|dz|^2}{(1-|z|^2)^2}$ on D). Show that the Möbius maps for which $\varphi(D) = D$ are:*[1]

$$\varphi(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

with $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. So:[2]

$$\text{Möb}(D) = \left\{\frac{az+b}{\bar{b}z+\bar{a}} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\right\} \cong \{\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right) : a, b \in \mathbb{C}, \det = 1\}/\{e^{i\theta}\text{Id}\} = PSU(1,1)$$

[1]You could repeat the proof for $\mathbb{H}$ for D, so asking yourself which Möbius maps send ∂D to ∂D. The shortcut is to observe that isometries $D \to D$ arise from $D \to \mathbb{H} \to \mathbb{H} \to D$ where $\mathbb{H} \to \mathbb{H}$ are the isometries we found above, and the maps $D \to \mathbb{H}$ (and back) are the isometries τ, τ^{-1} .

You can calculate compositions of Möbius maps by multiplying the corresponding matrices.

[2]For $SU(2)$ the $(2,1)$ entry of the matrix would need to be $-\bar{b}$, so that $\det = |a|^2 + |b|^2$. For $SU(1,1)$ the signature of the quadratic form has one + and one − sign: $\det = +|a|^2 - |b|^2$.

Notice that $a \neq 0$, so you can rescale numerator and denominator so that $|a| = 1$, so you can replace $a = e^{i\theta/2}$. Then φ becomes:

$$\varphi(z) = e^{i\theta}\,\frac{z+b}{\bar{b}z+1}.$$

with $b \in \mathbb{C}$ and $|b| < 1$. In particular, then $\varphi(0) = e^{i\theta}b$ so picking $b > 0 \in \mathbb{R}$ shows that there is a Möbius map with $\varphi(D) = D$ and $\varphi(0) =$ some chosen point in D.

Isometries of the hyperbolic disc D and the hyperbolic plane $\mathbb{H}$

Theorem *The group of orientation-preserving isometries of $\mathbb{H}$ contains Möb($\mathbb{H}$). The group of all isometries of $\mathbb{H}$ contains Möb($\mathbb{H}$) and the reflection $z \mapsto -\overline{z}$ (so it contains the orientation-reversing isometries $\psi(z) = \frac{-a\overline{z}+b}{-c\overline{z}+d}$).*

Remark. Later we show that there are no other isometries.

Proof. We start by checking that Möb($\mathbb{H}$) are isometries.

$$\frac{|dz|^2}{(\operatorname{Im} z)^2} = \frac{|d(\varphi(z))|^2}{(\operatorname{Im}\varphi(z))^2}.$$

First, $d\varphi(z) = \varphi'(z)\,dz$ where, having normalized: $ad - bc = 1$,

$$\varphi'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

Secondly,

$$\operatorname{Im}\varphi(z) = \operatorname{Im}\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \operatorname{Im}\frac{(ax+iay+b)(cx-icy+d)}{|cz+d|^2} = \frac{(ad-bc)\,y}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

The required equality above then follows.

For the last part, notice that $|d(-\overline{z})|^2 = (-d\overline{z})(-dz) = |dz|^2$, and $\operatorname{Im}(-\overline{z}) = \operatorname{Im}(z)$. □

Exercise. *Show that Möb(D) are orientation-preserving isometries of D, and that the reflection $z \mapsto \overline{z}$ is an orientation-reversing isometry of D. Notice in particular that the rotations $z \mapsto e^{i\theta}z$ are isometries of D, and that the reflection in the line with angle θ to the real axis is $z \mapsto e^{2i\theta}\overline{z} = e^{i\theta}\overline{e^{-i\theta}z}$ so also an (orientation-reversing) isometry.*

Now the issue is: how can we show that the above are all isometries? How can we be sure we have not omitted any? The easiest route to prove this, is to use geodesics, as follows.

REFERENCES

1. Boothby, William M. (1986), An introduction to differentiable manifolds and Riemannian geometry, Pure and Applied Mathematics, 120 (2nd ed.), Academic Press, ISBN 0121160521

2. Cartan, Élie (1983), Geometry of Riemannian Spaces, Math Sci Press, ISBN 978-0-915692-34-7; translated from 2nd edition of Leçons sur la géométrie des espaces de Riemann (1951) by James Glazebrook.

3. do Carmo, Manfredo P. (2016), Differential Geometry of Curves and Surfaces (revised & updated 2nd ed.), Mineola, NY: Dover Publications, Inc., ISBN 0-486-80699-5

4. do Carmo, Manfredo (1992), Riemannian geometry, Mathematics: Theory & Applications, Birkhäuser, ISBN 0-8176-3490-8

5. Gray, Alfred; Abbena, Elsa; Salamon, Simon (2006), Modern Differential Geometry of Curves And Surfaces With Mathematica®, Studies in Advanced Mathematics (3rd ed.), Boca Raton, FL: Chapman & Hall/CRC, ISBN 978-1-58488-448-4

6. Toponogov, Victor A. (2005), Differential Geometry of Curves and Surfaces: A Concise Guide, Springer-Verlag, ISBN 978-0-8176-4384-3

7. Valiron, Georges (1986), The Classical Differential Geometry of Curves and Surfaces, Math Sci Press, ISBN 978-0-915692-39-2 Full text of book

8. Warner, Frank W. (1983), Foundations of differentiable manifolds and Lie groups, Graduate Texts in Mathematics, 94, Springer, ISBN 0-387-90894-3

9. Geometry of Surfaces (Universitext) Corrected Edition by John Stillwell

10. Elements of Algebra: Geometry, Numbers, Equations (Undergraduate Texts in mathematics by John Stillwell

11. Differential Geometry of Curves and Surfaces by Thomas F. Banchoff, Stephen T. Lovett

12. Introduction to Topology: Third Edition (Dover Books on Mathematics) by Bert Mendelson