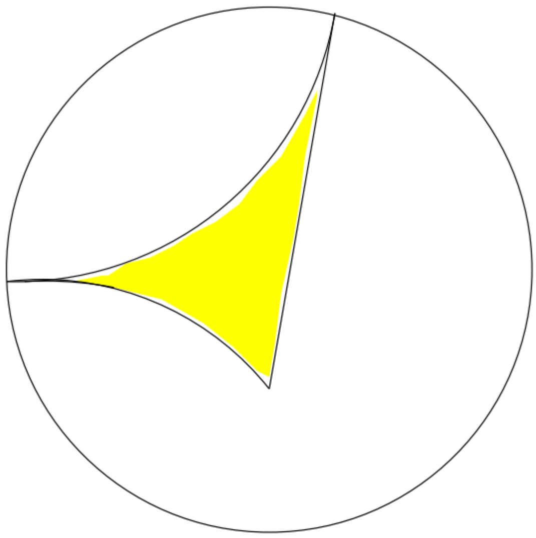
Hyperbolic triangles

A hyperbolic triangle Δ is given by three distinct points in H or D joined by geodesics. We see immediately from Gauss-Bonnet that the sum of the angles of a triangle is given by

$$A + B + C = \pi - \text{Area}(\Delta).$$

We can also consider hyperbolic triangles which have one or more vertices 'at infinity', i.e. on the boundary of H or D. These triangles are called *asymptotic*, *doubly (or bi-) asymptotic* and *triply (or tri-) asymptotic*, according to the number of vertices at infinity. The angle at a vertex at infinity is always 0, since all geodesics in H or D meet the boundary at right angles.



Theorem 5.1 (The cosine rule for hyperbolic triangles) *If Δ is a hyperbolic triangle in D with vertices at a, b, c and*

$$\alpha = d_D(b, c), \quad \beta = d_D(a, c) \text{ and } \gamma = d_D(a, b)$$

then

$$\cosh\gamma = \cosh\alpha \, \cosh\beta - \sinh\alpha \, \sinh\beta \, \cos\theta$$

where θ is the internal angle of Δ at c.

Proof: Because the group of isometries of D acts transitively on D we can assume that $c = 0$. Moreover, since the rotations $z \mapsto e^{i\phi}z$ are isometries which fix 0, we can also assume that a is real and positive. Then $\beta = 2\tanh^{-1}(a)$ so

$$a = \tanh(\beta/2)$$

and similarly

$$b = e^{i\theta}\tanh(\alpha/2)$$

while

$$\tanh(\gamma/2) = \left|\frac{b - a}{1 - \bar{a}b}\right|.$$

Recall that

$$\frac{1+\tanh^2(\gamma/2)}{1-\tanh^2(\gamma/2)} = \cosh(\gamma)$$

so

$$\cosh(\gamma) = \frac{|1-\bar{a}b|^2 + |b-a|^2}{|1-\bar{a}b|^2 - |b-a|^2} = \frac{(1+|a|^2)(1+|b|^2) - 2(\bar{a}b + a\bar{b})}{(1-|a|^2)(1-|b|^2)}.$$

Now

$$\frac{1+|a|^2}{1-|a|^2} = \frac{1+\tanh^2(\beta/2)}{1-\tanh^2(\beta/2)} = \cosh\beta$$

as above, and similarly

$$\frac{1+|b|^2}{1-|b|^2} = \cosh\alpha$$

while

$$\frac{2(\bar{a}b + a\bar{b})}{(1-|a|^2)(1-|b|^2)} = \frac{2\tanh(\alpha/2)\tanh(\beta/2)(e^{i\theta}+e^{-i\theta})}{\mathrm{sech}^2(\alpha/2)\mathrm{sech}^2(\beta/2)} = \sinh\alpha\ \sinh\beta\ \cos\theta.$$

This completes the proof. □

Theorem 5.2 *(*The sine rule for hyperbolic triangles) Let Δ be a hyperbolic triangle in D with internal angles A, B, C at vertices a, b, c and

$$\alpha = d_D(b,c), \quad \beta = d_D(a,c) \text{ and } \gamma = d_D(a,b).$$

Then

$$\frac{\sin A}{\sinh\alpha} = \frac{\sin B}{\sinh\beta} = \frac{\sin C}{\sinh\gamma}.$$

Proof: Two alternatives approaches:

1) Use the cosine rule to find an expression for $\sinh^2\alpha\ \sinh^2\beta\ \sin^2 C$ in terms of $\cosh\alpha$, $\cosh\beta$ and $\cosh\gamma$ which is symmetric in α, β and γ, and deduce that

$$\sinh^2\alpha\ \sinh^2\beta\ \sin^2 C = \sinh^2\alpha\ \sinh^2\gamma\ \sin^2 B = \sinh^2\gamma\ \sinh^2\beta\ \sin^2 A.$$

2) First prove that if $C = \pi/2$ then $\sin A \sinh\gamma = \sinh\alpha$ by applying the cosine rule to Δ in two different ways. Then deduce the result in general by dropping a perpendicular from one vertex of Δ to the opposite side. □

Gauss-Bonnet and its limits give the following:

Theorem 5.3 *(*Areas of hyperbolic triangles*)*

(i) The area of a triply asymptotic hyperbolic triangle Δ is π.

(ii) The area of a doubly asymptotic hyperbolic triangle Δ with internal angle θ is $\pi - \theta$.

(iii) The area of an asymptotic hyperbolic triangle Δ *with internal angles* θ *and* ϕ *is* $\pi - \theta - \phi$*.*

(iv) The area of a hyperbolic triangle Δ *with internal angles* θ*,* ϕ *and* ψ *is* $\pi - \theta - \phi - \psi$*.*

Non-Euclidean geometry

As we see above, the analogy between Euclidean geometry and its theorems and the geometry of the hyperbolic plane is very close, so long as we replace lines by geodesics, and Euclidean isometries (translations, rotations and reflections) by the isometries of H or D. In fact it played an important historical role.

For centuries, Euclid's deduction of geometrical theorems from self-evident common notions and postulates was thought not only to represent a model of the physical space in which we live, but also some absolute logical structure. One postulate caused some problems though – was it really self-evident? Did it follow from the other axioms? This is how Euclid phrased it:

"That if a straight line falling on two straight lines makes the interior angle on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which the angles are less than two right angles".

Some early commentators of Euclid's *Elements*, like Posidonius (1st Century BC), Geminus (1st Century BC), Ptolemy (2nd Century AD), Proclus (410 - 485) all felt that the parallel postulate was not sufficiently evident to accept without proof.

Here is a page from a medieval edition of Euclid dating from the year 888. It is handwritten in Greek. The manuscript, contained in the Bodleian Library, is one of the earliest surviving editions of Euclid.

The controversy went on and on with Greek and Islamic mathematicians puzzling over it. In 1621 Sir Henry Savile, Warden of Merton College, called attention to two blemishes in Euclidean geometry: the theory of parallels and the theory of proportion (nevertheless you can see Euclid, wearing an unsuitable hat, standing next to him on the memorial in Merton College Chapel). Johann Lambert (1728-1777) realized that if the parallel postulate did not hold then the angles of a triangle add up to less than 180°, and that the deficit was the area. He found this worrying in many ways, not least because it says that there is an absolute scale – no distinction between similar and congruent triangles. Finally Janos Bolyai (1802-1860) and Nikolai Lobachevsky (1793-1856) discovered non-Euclidean geometry simultaneously. It satisfies all of Euclid's axioms except the parallel postulate, and we shall see that it is the geometry of H or D that we have been studying.

Bolyai became interested in the theory of parallel lines under the influence of his father Farkas, who devoted considerable energy towards finding a proof of the parallel postulate without success. He even wrote to his son:

"I entreat you, leave the doctrine of parallel lines alone; you should fear it like a sensual passion; it will deprive you of health, leisure and peace – it will destroy all joy in your life."

Another relevant figure in the discovery was Carl Friedrich Gauss (1777-1855), who as we have seen developed the differential geometry of surfaces. He was the first to consider the possibility of a geometry denying the parallel postulate. However, for fear of being ridiculed he kept his work unpublished, or maybe he never made the connection with the curvature of real world surfaces and the Platonic ideal of axiomatic geometry. Anyway, when he read Janos Bolyai's work he wrote to Janos's father:

"If I commenced by saying that I must not praise this work you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years."

Euclid's axioms were made rigorous by Hilbert. They begin with undefined concepts of

- "point"
- "line"
- "lie on" (a point lies on a line)
- "betweenness"
- "congruence of pairs of points"
- "congruence of pairs of angles".

Euclidean geometry is then determined by logical deduction from the following axioms:

EUCLID'S AXIOMS

I. AXIOMS OF INCIDENCE

1. Two points have one and only one straight line in common.
2. Every straight line contains a least two points.
3. There are at least three points not lying on the same straight line.

II. AXIOMS OF ORDER

1. Of any three points on a straight line, one and only one lies between the other two.

2. If A and B are two points there is at least one point C such that B lies between A and C.

3. Any straight line intersecting a side of a triangle either passes through the opposite vertex or intersects a second side.

III. AXIOMS OF CONGRUENCE

1. On a straight line a given segment can be laid off on either side of a given point (the segment thus constructed is congruent to the give segment).

2. If two segments are congruent to a third segment, then they are congruent to each other.

3. If AB and $A'B'$ are two congruent segments and if the points C and C' lying on AB and $A'B'$ respectively are such that one of the segments into which AB is divided by C is congruent to one of the segments into which $A'B'$ is divided by C', then the other segment of AB is also congruent to the other segment of $A'B'$.

4. A given angle can be laid off in one and only one way on either side of a given half-line; (the angle thus drawn is congruent to the given angle).

5. If two sides of a triangle are equal respectively to two sides of another triangle, and if the included angles are equal, the triangles are congruent.

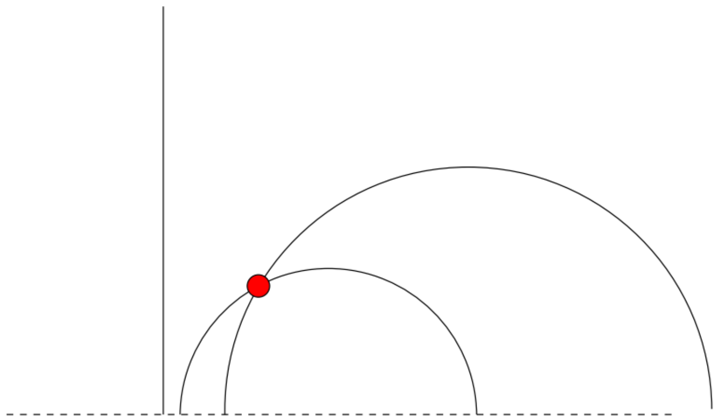
IV. AXIOM OF PARALLELS

Through any point not lying on a straight line there passes one and only one straight line that does not intersect the given line.

V. AXIOM OF CONTINUITY

1. If AB and CD are any two segments, then there exists on the line AB a number of points $A_1, \ldots, A_n$ such that the segments $AA_1, A_1A_2, \ldots, A_{n-1}A_n$ are congruent to CD and such that B lies between A and A_n

Clearly H does not satisfy the Axiom of Parallels:



The fact that hyperbolic geometry satisfies all the axioms except the parallel postulate is now only of historic significance and the reader is invited to do all the checking.

Often one model is easier than another. Congruence should be defined through the action of the group of isometries.

When are two surfaces different?

Homeomorphisms, diffeomorphisms, biholomorphisms

The class of surfaces affects when we want to view two surfaces as being the same or different. You have seen this in mathematics before: we think of two sets as "being the same" (**isomorphic**) if there is a bijection $: S_1 \to S_2$ between them, whereas for vector spaces (sets with additional structures called addition and rescaling) we want the bijection to preserve the additional structures, so we want a bijection f with f, f^{-1} both *linear*. For our four classes of surfaces, we define:

(1) Two topological surfaces are isomorphic if they are **homeomorphic**.
Explicitly: $f : S_1 \to S_2$ is a bijection, and f, f^{-1} are continuous.
(2) Two smooth surfaces in $\mathbb{R}^3$ are isomorphic if they are **diffeomorphic**.
Explicitly: $f : S_1 \to S_2$ is a bijection, and f, f^{-1} are smooth.[1]
(3) Two abstract smooth surfaces are isomorphic if they are **diffeomorphic**.
(4) Two Riemann surfaces are isomorphic if they are **biholomorphic**,
Explicitly: $f : S_1 \to S_2$ is a bijection, and f, f^{-1} are holomorphic.

Notice that a diffeomorphism/biholomorphism is in particular also a homeomorphism, so the underlying topological surfaces are the same. However, for all we know, there may be several ways to turn a topological surface into a smooth surface or a Riemann surface, and these different ways may not be related by a diffeomorphism/biholomorphism even though the surfaces are homeomorphic. We now check what happens for tori.

[1]Since surfaces in $\mathbb{R}^3$ are an abstract surface S together with the additional structure of an embedding $S \to \mathbb{R}^3$, a more appropriate notion of isomorphic is actually **isotopy**: a smooth family of embeddings. Explicitly: a smooth map $H : S \times [0, 1] \to \mathbb{R}^3$ such that $H(\cdot, 0) : S \to \mathbb{R}^3$, $H(\cdot, 1) : S \to \mathbb{R}^3$ are the two embedded surfaces, and we want $H(\cdot, t) : S \to \mathbb{R}^3$ to be an embedding for each $t \in [0, 1]$.

Classification of tori

Definition (Torus). *A torus is any topological space X which is homeomorphic to $S^1 \times S^1$ (using the product topology).*

There are several (homeomorphic) ways to describe S^1 as a topological space:

(1) $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ with the subspace topology,
(2) $S^1 = \mathbb{R}/\mathbb{Z}$ with the quotient topology, where we identify $x \sim x + n$ any $n \in \mathbb{Z}$,
(3) $S^1 = [0,1]/(0 \sim 1)$ with the quotient topology.

For example, a homeomorphism from (2) to (1) is $x \mapsto e^{2\pi i x}$.

The torus viewed as the square by identifying parallel sides arises from description (3) of S^1; the torus as a quotient $\mathbb{R}^2/\mathbb{Z}^2$ by the translation group $\mathbb{Z}^2$ arises from description (2); and the torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ arises naturally from description (1).

As a topological surface, all tori are the same: homeomorphic to $S^1 \times S^1$. But how many different smooth surfaces are homeomorphic to $S^1 \times S^1$, and how many different Riemann surfaces are homeomorphic to $S^1 \times S^1$? We will not prove the following hard theorem (a consequence of the classification of compact surfaces):

Theorem *Any smooth surface which is topologically a torus is diffeomorphic to $S^1 \times S^1$.*

This turns out to be false for Riemann surfaces: there are many non-biholomorphic Riemann surfaces which are tori. Again, we will not prove the following hard theorem:

Theorem (Elliptic curves over $\mathbb{C}$). *Any Riemann surface which is topologically a torus is biholomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ which we may rescale so that $\Lambda = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$ with $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. These are called the* ***elliptic curves****.*

We will now show explicitly why two such tori $\mathbb{C}/\Lambda_1, \mathbb{C}/\Lambda_2$ are diffeomorphic, and we will show that they are not always biholomorphic.

To show that they are diffeomorphic, we might as well show that all quotients $\mathbb{C}/\Lambda$ are diffeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$ (which is the case $\tau = i$), then $\mathbb{C}/\Lambda_1 \cong \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{C}/\Lambda_2$ are diffeomorphic, as required. Identifying $\mathbb{C} = \mathbb{R}^2$, $z \equiv x + iy$, write $\tau = a + ib$. Matrix multiplication

$$\left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$$

is of course a smooth map (it is linear!) and it is bijective (the determinant $b = \mathrm{Im}(\tau) > 0$ is non-zero), and it maps $\mathbb{Z}^2 \to \Lambda$ bijectively (the columns are the images of the standard basis). Thus it defines a diffeomorphism $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}/\Lambda$. The map is however not holomorphic.[1]

More generally, suppose we are given a biholomorphism

$$f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda' \quad \text{where } \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad \Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'.$$

This means that[2] it arises from quotienting a holomorphic map

$$\widetilde{f} : \mathbb{C} \to \mathbb{C} \text{ with } \widetilde{f}(\Lambda) \subset \Lambda'$$

which is injective on any $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$-translate of the unit square $(0,1) \times i(0,1) \subset \mathbb{C}$. It follows that $\widetilde{f}$ maps Λ bijectively to Λ'. It easily follows that $\widetilde{f}$ grows linearly in $z \in \mathbb{C}$. So the Taylor series of $\widetilde{f}$ does not contain order z^2 or higher terms. So $\widetilde{f}(z) = Az + B$ for some $A \in \mathbb{C} \setminus \{0\}, B \in \mathbb{C}$. Taking $z = 0$ shows that $B \in \Lambda'$, so by composing $\widetilde{f}$ with the translation

[1] $f(x + iy) = (x + ay) + iby$ has $\partial_x f = 1$ but $\partial_y f = a + ib$, so the Cauchy-Riemann equations $\partial_x f = -i\,\partial_y f$ fail.

[2] Strictly speaking, we only know this locally, because we used the quotient $\mathbb{C} \to \mathbb{C}/\text{Lattice}$ to define the holomorphic local parametrizations. However, by the **Identity theorem** from complex analysis you know that you can patch together the local Taylor series uniquely to obtain a global holomorphic map defined on $\mathbb{C}$.

$z \mapsto z - B$ we may as well assume that $B = 0$. So $\widetilde{f}(z) = Az$ is linear. Thus the problem reduces to classifying lattices $\Lambda \subset \mathbb{C}$ up to $\mathbb{C}$-linear bijections!

Since $A\omega_1, A\omega_2$ is required to be a $\mathbb{Z}$-linear basis for Λ', the two bases $A\omega_1, A\omega_2$ and ω_1', ω_2' of Λ' differ by a $\mathbb{Z}$-linear bijection (think of row-reduction but working over $\mathbb{Z}$). So

$$\omega_1' = aA\omega_1 + bA\omega_2 \qquad \omega_2' = cA\omega_1 + dA\omega_2.$$

for some invertible integer-valued matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$. Thus, using the convention that $\tau = \pm\frac{\omega_1}{\omega_2}$ adjusting the sign so that $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$,

$$\tau' = \pm\frac{\omega_1'}{\omega_2'} = \pm\frac{aA\omega_1 + bA\omega_2}{cA\omega_1 + dA\omega_2} = \pm\frac{\pm a\tau + b}{\pm c\tau + d}.$$

So the matrix acts on the τ parameter like a Möbius map. By properties of Möbius maps[1] since $\tau, \tau' \in \mathbb{H}$, we deduce that $\tau' = M \cdot \tau$ where $M = \pm \left(\begin{smallmatrix} \pm a & b \\ \pm c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{Z})$.

Corollary $\mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau) \cong \mathbb{C}/(\mathbb{Z}1 + \mathbb{Z}\tau')$ *are biholomorphic if and only if* $\tau, \tau' \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *lie in the same orbit of the* $PSL(2, \mathbb{Z})$*-action on* $\mathbb{H}$ *by Möbius maps.*

Corollary *Riemann surfaces which are topologically a torus are classified up to biholomorphism by* $[\tau] \in \mathbb{H}/PSL(2, \mathbb{Z})$.

Cultural remark: this **moduli space**, $\mathbb{H}/PSL(2, \mathbb{Z})$, of modular parameters $[\tau]$ is in fact itself a Riemann surface biholomorphic to $\mathbb{C}$.

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