Complex analysis and the hyperbolic plane

The intricate metric structure of the hyperbolic plane – geodesics, triangles and all – is actually determined purely by the holomorphic functions on it, so we could also think of hyperbolic geometry as a branch of complex analysis. Here is the theorem that makes it work:

Theorem 5.4 *Any holomorphic homeomorphism $f : D \to D$ is an isometry of the hyperbolic metric.*

Proof: The argument follows Schwartz's lemma. By applying an isometry we can assume that $f(0) = 0$, and since the image of f is D, we have $|f(z)| < 1$ if $z \in D$. Now since $f(0) = 0$, $f_1(z) = f(z)/z$ is holomorphic and applying the maximum principle to a disc of radius $r < 1$ we get

$$|f_1(z)| \leq \frac{1}{r}$$

and in the limit as $r \to 1$, $|f_1(z)| \leq 1$ or equivalently

$$|f(z)| \leq |z|.$$

Since f is a homeomorphism, its inverse satisfies the same inequality so

$$|z| \leq |f(z)|$$

and $|f_1(z)| = 1$ everywhere. Since this is true at an interior point the function must be a constant c so $f(z) = cz$ and since $|f(z)| = |z|$,

$$f(z) = e^{i\theta} z$$

which is an isometry. □

In fact a similar result holds for **C**:

Theorem 5.5 *Any holomorphic homeomorphism $f : \mathbf{C} \to \mathbf{C}$ is of the form $f(z) = az + b$ with $a \neq 0$.*

If $|a| = 1$ this is an isometry of the Euclidean metric $dx^2 + dy^2$. The extra scaling $z \mapsto \lambda z$ is what gives rise in classical geometrical terms to similar but non-congruent triangles.

Proof: For $|z| > R$ consider the function $g(z) = f(1/z)$. Suppose g has an essential singularity at $z = 0$. Then the Casorati-Weierstrass theorem (Exercise 17.5 in [3]) tells us that $g(z)$ gets arbitrarily close to any complex number if z is small enough, and in particular to values in the image of $\{z : |z| \leq R\}$ under f. But we assumed f was bijective, which is a contradiction. It follows that g has at most a pole at infinity and so $f(z)$ must be a polynomial of some degree k.

However the equation $f(z) = c$ then has k solutions for most values of c, and again since f is bijective we must have $k = 1$ and

$$f(z) = az + b.$$

□

For completeness, we add the following

Theorem 5.6 *Any holomorphic homeomorphism f of the Riemann sphere to itself is a Möbius transformation $z \mapsto (az + b)/(cz + d)$.*

Proof: By using a Möbius transformation we can assume that $f(\infty) = \infty$ and then the previous theorem tells us that $f(z) = az + b$. □

These results are all about the complex plane and its subsets. In fact hyperbolic geometry has an important role to play in the study of compact Riemann surfaces. Recall that local holomorphic coordinates on a Riemann surface are related by holomorphic transformations and these preserve angles. Given two smooth curves on a Riemann surface, it makes good sense to define their angle of intersection and this is called a *conformal structure*. A metric also defines angles so we can consider metrics compatible with the conformal structure of a Riemann surface. In a local coordinate z such a metric is of the form

$$f dz d\bar{z} = f(dx^2 + dy^2).$$

The remarkable result is the following *uniformization* theorem:

Theorem 5.7 *Every closed Riemann surface X has a metric of constant Gaussian curvature compatible with its conformal structure.*

Note that by the Gauss-Bonnet theorem $K > 0$ implies $\chi(X) > 0$, i.e. X is a sphere, $K = 0$ implies $\chi(X) = 0$, i.e. X is a torus, and $K < 0$ gives $\chi(X) < 0$.

Proof: The proof is a corollary of a difficult theorem called the Riemann mapping theorem. Recall that a space is *simply-connected* if it is connected and every closed path can be shrunk to a point. The Riemann mapping theorem (proved by Poincaré and Koebe) says that every simply-connected Riemann surface is holomorphically homeomorphic to either the Riemann sphere, $\mathbf{C}$ or H.

If X is any reasonable topological space, one can form its *universal covering space* $\tilde{X}$ (see [2]) which is simply connected and has

- a projection $p : \tilde{X} \to X$
- every point $x \in X$ has a neighbourhood V such that $p^{-1}(V)$ consists of a disjoint union of open sets each of which is homeomorphic to V by p
- there is a group π of homeomorphisms of $\tilde{X}$ such that $p(gy) = p(y)$, so that π permutes the different sheets in $p^{-1}(V)$.
- no element of π apart from the identity has a fixed point
- X can be identified with the space of orbits of π acting on $\tilde{X}$.

The standard example of this is $X = S^1$, $\tilde{X} = \mathbf{R}$, $p(t) = e^{it}$ and $\pi = \mathbf{Z}$ acting by $t \mapsto t + 2n\pi$. It is easy to see that the universal covering of a Riemann surface is a Riemann surface, so applying the Riemann mapping theorem we see that $\tilde{X}$ is either the Riemann sphere, $\mathbf{C}$ or H.

So consider the cases:

- If $\tilde{X}$ is the sphere S, it is compact and so $p : \tilde{X} \mapsto X$ has only a finite number k of sheets. By counting vertices, edges and faces it is clear that $\chi(\tilde{X}) = k\chi(X)$. Since $\chi(S) = 2$, we must have $k = 1$ or 2, but if the latter $\chi(X) = 1$ which is not of the allowable form $2-2g$ for an orientable surface and a Riemann surface *is* orientable. So it is only the Riemann sphere in this case.
- If $\tilde{X} = \mathbf{C}$, we appeal to Theorem 5.5. The group π of covering transformations is holomorphic and so each element is of the form $z \mapsto az + b$. But π has no fixed points, so $az + b = z$ has no solution which means that $a = 1$. the transformations $z \mapsto z + b$ are just translations and are isometries of the metric $dx^2 + dy^2$ which has $K = 0$.

- If $\tilde{X} = H$, then from Theorem 5.4, the action of π preserves the hyperbolic metric.

□

So we see that these abstract metrics have a role to play in the study of Riemann surfaces – a long long way from surfaces in $\mathbf{R}^3$.

The Euler characteristic

Euler characteristic of regular polyhedra

Notice the following pattern in the number of vertices, edges, faces of the Platonic solids:

Regular polyhedron	Face type	V	E	F	$\chi = V - E + F$
Tetrahedron	Triangle	4	6	4	2
Cube	Square	8	12	6	2
Octahedron	Triangle	6	12	8	2
Dodecahedron	Pentagon	20	30	12	2
Icosahedron	Triangle	12	30	20	2

The alternating difference $\chi = V - E + F$ is called the **Euler characteristic**.

Why is it always 2 for Platonic solids?

The reason is that χ is a **topological invariant** of the topological surface, meaning it is a quantity which is the same for any two surfaces which are homeomorphic. Since all Platonic solids are homeomorphic to the sphere, they all have $\chi = \chi(S^2) = 2$.

The homeomorphism between the Platonic solid and the sphere defines a **cellular decomposition** of the sphere, which is a subdivision of the sphere into regions homeomorphic to discs (in this case, curved polygons).

[1]Recall **Möbius maps** are the biholomorphisms $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1 \to \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. These maps don't change if you rescale all a, b, c, d by the same non-zero complex number, so you may arrange that $ad - bc = 1$ (which leaves the freedom of rescaling all by ± 1). So such maps are parametrized by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})/(\pm I) = PSL(2, \mathbb{C})$, and indeed the maps compose according to matrix multiplication. So the group of Möbius maps is isomorphic to $PSL(2, \mathbb{C})$. The subgroup of Möbius maps which send $\mathbb{H} \to \mathbb{H}$ is in fact $PSL(2, \mathbb{R})$ (Exercise: first force $\mathbb{R} \mapsto \mathbb{R}$, then you just need to ensure the maps don't flip $\mathbb{H}$ to $-\mathbb{H}$)).

Cellular decomposition

Definition 4.1 (Cellular decomposition)**.** *A* ***cellular decomposition*** *of a topological surface S is a collection of continuous maps, called* ***cells****,*

$$v_i : \mathbb{D}^0 \to S \qquad e_j : \mathbb{D}^1 \to S \qquad f_k : \mathbb{D}^2 \to S$$

respectively called 0-cells, 1-cells, 2-cells, where[1] $\mathbb{D}^n = \{p \in \mathbb{R}^n : \|p\| \leq 1\}$ *is the n-dimensional unit disc, and we require that:*

1. *each map restricted to the interior of the disc is a homeomorphism onto the image,*[2]
2. *the boundary of the disc is mapped into the image of the lower-dimensional cells,*[3]
3. *S is partitioned by the 0-cells and the interiors of the higher dimensional cells.*[4]

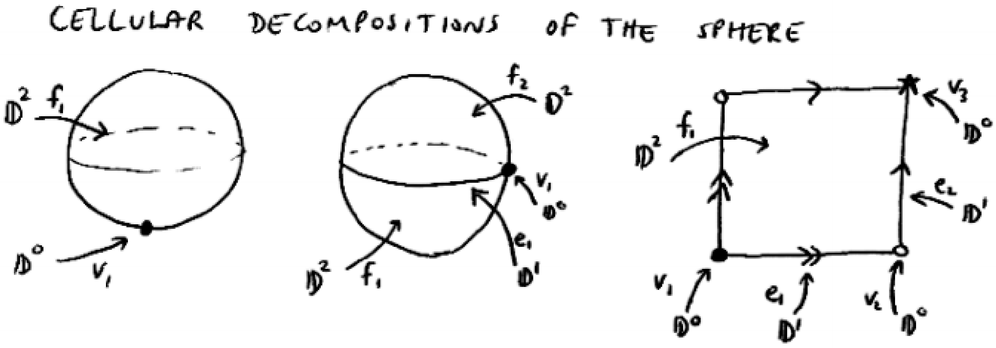
Remarks.

◇ The maps restricted to the boundary $\partial\mathbb{D}^n$ are called **attaching maps** (can be non-injective).

◇ Notice you are building the space inductively by dimension: you start with a bunch of points $X^0 = \bigsqcup v_i(\mathbb{D}^0)$, then you attach line segments $X^1 = X^0 \cup \bigcup e_j(\mathbb{D}^1)$ where the attaching maps $e_j|_{\{0,1\}}$ land inside X^0, and finally you attach 2-discs $X^2 = X^1 \cup \bigcup f_k(\mathbb{D}^2) = S$ where the attaching maps $f_k|_{S^1}$ land in X^1. The subspace $X^i \subset S$ is called the i-**skeleton**.

⋄ Condition (3) ensures there are no redundancies or silly overlaps: the vertices are distinct, no vertices touch the interior of an edge or face, no edge touches the interior of a face.
⋄ The above definition works more generally for any n-manifold M, in which case you can have cells $c_i : \mathbb{D}^{d_i} \to M$ of any dimension $d_i \in \{0, 1, 2, \ldots, n\}$.
⋄ A **triangulation** is a cellular decomposition, where the faces are identified (via homeomorphisms) with triangles and the attaching maps are all injective, so that the boundary edges of the triangles are precisely the 1-cells, and the boundary vertices of the edges are precisely the 0-cells. These conditions are quite harsh, so it is usually very messy[5] to triangulate a surface.

Example. Here are three examples of cellular decompositions of S^2:



Here $f_1 : \mathbb{D}^2 \cong$ (square)$\to$ (square with identifications), and this map is a homeomorphism on the interior, but on the boundary it is not injective. Notice that (just as for the Platonic solids, which also yield cellular decompositions of S^2) the alternating sum of the numbers of cells is always 2:

$$1 - 0 + 1 = 2 \qquad 1 - 1 + 2 = 2 \qquad 3 - 2 + 1 = 2.$$

---

[1]So $\mathbb{D}^0 = \{\text{point}\}$, $\mathbb{D}^1 = [0,1] \subset \mathbb{R}$ and $\mathbb{D}^2 = \mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Their interiors are $\mathrm{Int}(\mathbb{D}^0) = \emptyset$, $\mathrm{Int}[0,1] = (0,1)$ and $\mathrm{Int}(\mathbb{D}^2) = D = \{z \in \mathbb{R}^2 : \|z\| < 1\}$. Their boundaries are $\partial\mathbb{D}^0 = \emptyset$, $\partial\mathbb{D}^1 = \{0\} \cup \{1\}$ and $\partial\mathbb{D}^2 = S^1$.

[2]$e_j : (0,1) \to e_j((0,1)) \subset S$, $f_k : D \to f_k(D) \subset S$ are homeomorphisms (no condition on v_i as $\mathrm{Int}\mathbb{D}^0 = \emptyset$).

[3]$e_j(0), e_j(1) \in \bigcup v_i(\mathbb{D}^0)$ and $f_k(S^1) \subset \bigcup v_i(\mathbb{D}^0) \cup \bigcup e_j(\mathbb{D}^1)$.

[4]$S = \bigsqcup v_i(\mathbb{D}^0) \sqcup \bigsqcup e_j(\mathrm{Int}\,\mathbb{D}^1) \sqcup \bigsqcup f_k(\mathrm{Int}\,\mathbb{D}^2)$ is a disjoint union of subsets.

[5]try to triangulate the torus, viewed as a square with parallel sides identified.

A very general (hard) fact from algebraic topology is:

Theorem 4.2. *Any topological manifold M (e.g. a topological surface) admits a cellular decomposition. If there are finitely many cells, the alternating sum of the numbers of cells*

$$\chi(M) = (\#0\text{-cells}) - (\#1\text{-cells}) + (\#2\text{-cells}) - (\#3\text{-cells}) + \cdots$$

is the same for any cellular decomposition. It is called the ***Euler characteristic*** *of M.*

Corollary 4.3. *If M, N are homeomorphic topological manifolds then $\chi(M) = \chi(N)$. So the Euler characteristic is a topological invariant.*

Proof. If $f : M \to N$ is a homeomorphism, then a cellular decomposition $c_i : \mathbb{D}^{d_i} \to M$ of M determines a cellular decomposition $f \circ c_i : \mathbb{D}^{d_i} \to N$ of N. So $\chi(M) = \sum(-1)^{d_i} = \chi(N)$. □

Example. We obtained the torus from a square by identifying the parallel edges. The whole square is a 2-cell $f_2 : \mathbb{D}^2 \to T^2$, the two non-parallel edges are two 1-cells $e_1, e_2 : \mathbb{D}^1 \to T^2$, and the four vertices of the square are identified with one 0-cell $v_1 : \mathbb{D}^0 \to T^2$. Thus

$$\chi(T^2) = 1 - 2 + 1 = 0.$$

For $\mathbb{R}P^2$ the four vertices instead define two 0-cells, so

$$\chi(\mathbb{R}P^2) = 2 - 2 + 1 = 1.$$

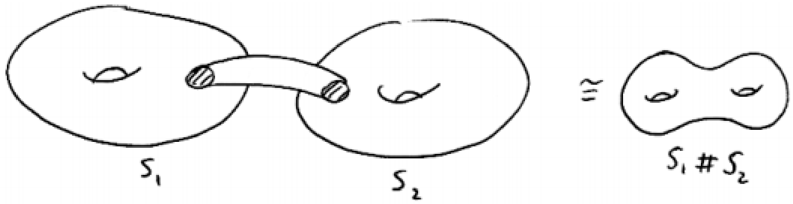
Connected sum

In general, given two surfaces S_1 and S_2, we can form two new surfaces:

(1) the disjoint union: $S_1 \sqcup S_2$,
(2) the connected sum: $S_1 \# S_2$.

Easy exercise. Show that any connected component of a surface is a surface. Deduce that any surface equals a disjoint union of connected surfaces.

The **connected sum** $S_1 \# S_2$ is obtained by removing a "disc" from each of the two surfaces and identifying the circular boundaries.



This identification is the same (up to homeomorphism) as attaching a cylinder by gluing the two boundaries of the cylinder onto the two boundaries of the removed discs.

Exercise. Check that $S_1 \# S_2$ is indeed a topological surface. Convince yourself that if S_1, S_2 are connected then, up to homeomorphism, it does not matter which "discs" you pick.

Exercise. Connected sum with a sphere does nothing.

Connected sum with a torus is the same as attaching a handle

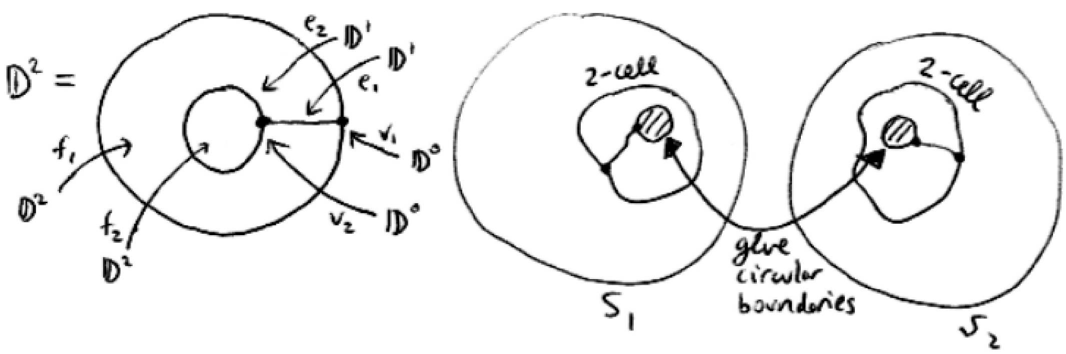
Additivity of the Euler characteristic

Lemma

(1) $\chi(S_1 \sqcup S_2) = \chi(S_1) + \chi(S_2)$,
(2) $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

Proof. [1] Pick a cellular decomposition of S_1, S_2. Then this defines a cellular decomposition of $S_1 \sqcup S_2$ so (1) follows immediately. The idea in (2) is that we remove two faces, which makes χ drop by 2, and we identify the circular boundaries so we lose one copy of S^1, which does not matter for χ since $\chi(S^1) = 0$ (since S^1 is a point with an interval attached to the point, so $\chi = 1 - 1 = 0$). This idea is correct if you triangulate S_1, S_2 and remove two triangular faces. However, if we instead want to work with cellular decompositions (which arise more naturally than triangulations) then the rigorous proof is a little more involved, as follows.

The surface $S_1 \# S_2$ up to homeomorphism only depends on the the choice of connected components in S_1 and in S_2 where you pick the "discs". So we might as well pick each "disc" in the interior of a 2-cell in the cellular decomposition. To do this without destroying the cellular decomposition[2] we subdivide each original 2-cell $f_0 : \mathbb{D}^2 \to S_i$ as follows:



The new edges e_1, e_2 map injectively into S_i since the original f_0 is injective on $\mathrm{Int}(\mathbb{D}^2)$, and similarly the new faces f_1, f_2 are injective on $\mathrm{Int}(\mathbb{D}^2)$. However, a comment is required about v_1. If $f_0(S^1)$ already contains a 0-cell, then we can use that for v_1, and χ will not have changed.[3] If $f_0(S^1)$ does not contain a 0-cell then, by the partitioning condition, $f_0(S^1)$ consists of images of edges, so creating a new v_1 means subdividing an edge into two. So we are also creating a new edge. So this new vertex/edge pair does not affect χ: $1 - 1 = 0$. This invariance of χ is a special instance of the very general invariance Theorem 4.2.

Next, we remove the small faces (f_2 in the picture) from S_1, S_2, which makes χ drop by 2. Finally we identify the two boundaries of those faces which means we identify the copies in S_1, S_2 of v_2, e_2 in the above picture, so χ does not change $(+1 - 1 = 0)$. So (2) follows. □

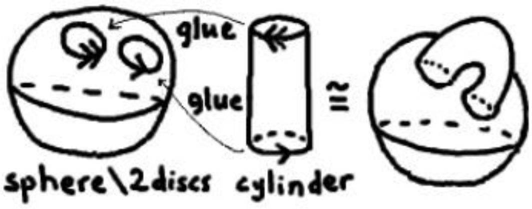
Remark. *If a topological space $S = A \cup B$ (such as a surface) is a union of two closed subsets, and suppose[4] S admits a cellular decomposition such that it induces cellular decompositions for $A \cap B, A, B$, then by counting you deduce $\chi(S) = \chi(A) + \chi(B) - \chi(A \cap B)$. Can you see how to use this to prove the above formula for $\chi(S_1 \# S_2)$?*

Attaching handles to a sphere

Observe that there is a natural way to orient the boundary of a "disc"[5] in the sphere: we ask that it obeys the **right-hand rule**[6], with the thumb pointing in the normal outward direction (so for very small discs, the boundary is oriented anti-clockwise if you are looking at the sphere $S^2 \subset \mathbb{R}^3$ from far away).

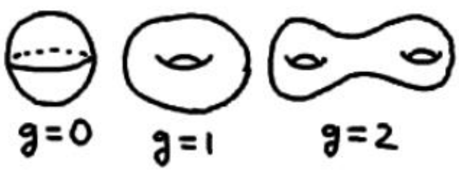
A cylinder is a space homeomorphic to $[0,1] \times S^1$. The boundaries are oriented as follows: $\{1\} \times S^1$ is oriented clockwise, $\{0\} \times S^1$ is oriented anticlockwise.

Attaching a **handle** to S^2 means you remove two disjoint "discs" from S^2, and you glue the two boundary circles of the cylinder $[0,1] \times S^1$ onto the two boundaries of the discs you removed in a way which preserves the above orientations (in practice: draw arrows on the circular boundaries, and glue in a way that respects the arrow directions). The orientation choices ensure that we can think of the handle as attached onto $S^2 \subset \mathbb{R}^3$ from the "outside":



Thus, starting from a sphere, we obtain a sequence of surfaces:

The number g of handles attached to S^2 is called the **genus of the surface**, and corresponds to the number of "doughnut holes":



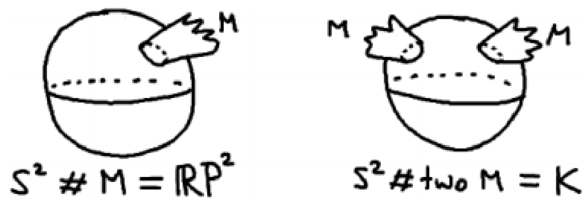
Lemma *Attaching a handle decreases χ by 2.*

Proof. To obtain a cellular decomposition of a cylinder $S^1 \times [0,1]$, we declare that $e_1 = [0,1] \cong \{1\} \times [0,1] \subset S^1 \times [0,1]$ is a 1-cell. View each of the circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$ as 1-cells e_2, e_3 which have been attached by identifying both endpoints to the same point, namely the endpoints $v_1 = (1,0)$, $v_2 = (1,1)$ of e_1. The cylinder itself defines a 2-cell bounded by e_1, e_2, e_3. Thus $\chi(\text{cylinder}) = 2 - 3 + 1 = 0$. When we attach the cylinder, we run a construction similar to the picture in the proof of Lemma 4.4. Namely, we remove two discs from the original surface (so a 2-cell), which makes χ drop by 2, whilst the identification of the boundary circles does not change χ (viewing the boundary circle as a 1-cell with both endpoints attached to the same 0-cell, we lose a 1-cell and a 0-cell, leaving χ unaffected). □

Attaching Möbius bands to a sphere

In Exercise sheet 1 you study Möbius bands. The Möbius band M is the quotient of the square $[0,1] \times [0,1]$ by identifying the vertical edges in opposite directions, $(0,y) \sim (1, 1-y)$. The boundaries $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ glue to give a circle.

Attaching a **Möbius band** to S^2 means you remove a "disc" from S^2, and you glue the boundary circle of M onto the boundary of the disc you removed. One cannot draw this in $\mathbb{R}^3$ without self-intersections, so schematically we will draw M as a wiggly cap:



The above are the first two of a sequence of surfaces one obtains by attaching Möbius bands to S^2

Lemma 4.6. *Attaching a Möbius band decreases χ by 1.*

Proof. This is similar to Lemma 4.5. M has a 2-cell (the square), three 1-cells (two of the four edges of the square are identified), two 0-cells (vertices are identified in pairs). So $\chi(M) = 0$. When we attach M, χ drops by 1 as we remove a disc (a 2-cell) from the original surface. □

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[1] **Non-examinable exercise.** *For n-manifolds M, N, explain how one constructs a connected sum $M\#N$ and show that $\chi(M\#N) = \chi(M) + \chi(N) - \chi(S^n)$, where S^n is the n-sphere.*

[2] making a hole inside the "disc" gives an annulus (up to homeomorphism), so it is no longer a 2-cell.

[3] before subdivision, f_0 contributes $+1$, after subdivision v_2, e_1, e_2, f_1, f_2 contribute $1 - 2 + 2 = 1$.

[4] More generally, this formula for χ holds whenever S is the union of the interiors of the closed sets A, B, as a consequence of the so-called *Mayer-Vietoris sequence* (see C3.1 Algebraic Topology).

[5] "Disc" will mean a continuous map $\mathbb{D}^2 \to S$ which is a homeomorphism onto its image.

[6] thumb pointing in the normal outward direction, index finger pointing in the oriented circular direction, and middle finger pointing towards the centre of the circle.

REFERENCES

1. Boothby, William M. (1986), An introduction to differentiable manifolds and Riemannian geometry, Pure and Applied Mathematics, 120 (2nd ed.), Academic Press, ISBN 0121160521

2. Cartan, Élie (1983), Geometry of Riemannian Spaces, Math Sci Press, ISBN 978-0-915692-34-7; translated from 2nd edition of Leçons sur la géométrie des espaces de Riemann (1951) by James Glazebrook.

3. do Carmo, Manfredo P. (2016), Differential Geometry of Curves and Surfaces (revised & updated 2nd ed.), Mineola, NY: Dover Publications, Inc., ISBN 0-486-80699-5

4. do Carmo, Manfredo (1992), Riemannian geometry, Mathematics: Theory & Applications, Birkhäuser, ISBN 0-8176-3490-8

5. Gray, Alfred; Abbena, Elsa; Salamon, Simon (2006), Modern Differential Geometry of Curves And Surfaces With Mathematica®, Studies in Advanced Mathematics (3rd ed.), Boca Raton, FL: Chapman & Hall/CRC, ISBN 978-1-58488-448-4

6. Toponogov, Victor A. (2005), Differential Geometry of Curves and Surfaces: A Concise Guide, Springer-Verlag, ISBN 978-0-8176-4384-3

7. Valiron, Georges (1986), The Classical Differential Geometry of Curves and Surfaces, Math Sci Press, ISBN 978-0-915692-39-2 Full text of book

8. Warner, Frank W. (1983), Foundations of differentiable manifolds and Lie groups, Graduate Texts in Mathematics, 94, Springer, ISBN 0-387-90894-3

9. Geometry of Surfaces (Universitext) Corrected Edition by John Stillwell

10. Elements of Algebra: Geometry, Numbers, Equations (Undergraduate Texts in mathematics by John Stillwell

11. Differential Geometry of Curves and Surfaces by Thomas F. Banchoff, Stephen T. Lovett

12. Introduction to Topology: Third Edition (Dover Books on Mathematics) by Bert Mendelson