MTH 611- GEOMETRY OF SURFACES AND ITS APPLICATION

EXAMINATION

INSTRUCTION TO STUDENTS: ATTEMPT **ANY FIVE** QUESTIONS

TIME ALLOCATED FOR THE EXAM: 3 HOURS

Question 1. (20 Marks)

Riemann surfaces arising from polynomial equations.

Briefly explain a natural way to make the sets

(1) $S_1 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)\} \cup \{+\infty\} \cup \{-\infty\}$
(2) $S_2 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)(z-3)\} \cup \{\infty\}$

into Riemann surfaces. Find homeomorphisms $S_1 \cong$ sphere, $S_2 \cong$ torus. *(Hints in footnote)*

Question 2. (20 Marks)

The Euler characteristic constrains graphs.

Given five points in the plane, show that it is impossible to connect each pair by paths which do not cross. Is it possible for five points in a torus?

Question 3. (20 Marks)

Holomorphic maps between Riemann surfaces.

Using the local form of a holomorphic map between Riemann surfaces, deduce:
Open mapping theorem: any holomorphic map $f : R \to S$ between Riemann surfaces, with R connected, is either constant or an open map, meaning f(any open set) is open.[1]

Deduce the following, for $f : R \to S$ holomorphic, R, S Riemann surfaces:

(1) If f is non-constant, R compact connected, then $f(R) \subset S$ is a connected component.
(2) If f is non-constant, R, S both compact connected, then f is surjective: $f(R) = S$.
(3) If R is compact connected, S non-compact connected, then f is constant.
(4) A holomorphic map $S \to \mathbb{C}$ on a compact connected Riemann surface is constant.
(5) Fundamental theorem of algebra: non-constant complex polynomials have a root.

[1] *Hint. Notice that to show a map is open, it's enough to show that for each p, there are some nice arbitrarily small open neighbourhoods of p which map to open sets.*

Footnote hint

Hints. It helps if you first ask yourself what local holomorphic coordinate you would use at solutions (z,w) of $w^2 = z$ (recall from lecture notes the discussion of the square root $z^{1/2}$). Then try to build the solution set S_1 by gluing two cut-domains: two copies of $\mathbb{C}$ cut from 1 to 2. Just like for Log z in lectures, each subset you cut gives rise to *two* copies of that subset in the Riemann surface. In order to be able to draw the Riemann surface inside $\mathbb{R}^3$, it is convenient to reflect one of the cut-domains about the x-axis. Near infinity, try using the coordinates $X = \frac{1}{z}$ and $Y = \frac{w}{z}$ instead of z, w, and ask yourself what happens for $X = 0$ (corresponding to "$z = \infty$"). For S_2 you will need a second cut, from 3 to ∞, and try instead $Y = \frac{w}{z^2}$.

Question 4. (20 Marks)

Implicit function theorem.

Consider $R = \{(z, w) \in \mathbb{C}^2 : w^3 = z^3 - z\}$. Use the implicit function theorem to check that R is a Riemann surface. Now consider the projection

$$\pi : R \to \mathbb{C},\ \pi(z, w) = z.$$

Find the branch points of π. Find the valency $v_\pi(p)$ at the ramification points.

Next, we seek how many points are "missing" at infinity. Write $z^3 - z = z^3(1 - z^{-2})$ for large $|z|$, and briefly explain that there are three holomorphic solution functions to $w^3 = z^3 - z$. Deduce that $\pi^{-1}(\{z \in \mathbb{C} : |z| > 100\})$ is biholomorphic to three punctured discs.

Compute the Euler characteristic of R using the Riemann-Hurwitz formula. Deduce that R is homeomorphic to a torus with three points removed.

Question 5. (20 Marks)

Riemann-Hurwitz formula.

In the following, all spaces are compact connected Riemann surfaces, and all maps are holomorphic maps. Deduce from the Riemann-Hurwitz formula that:

(1) if $f : R \to S$ is not constant, then the genus $g(R) \geq g(S)$.
(2) if $f : \mathbb{C}P^1 \to S$ is not constant, then S is homeomorphic to a sphere.
(3) if $f : R \to S$ has degree 1 then f is a biholomorphism.
(4) if R admits a meromorphic function with only one pole of order 1, then $R \cong \mathbb{C}P^1$.

Question 6. (20 Marks)

Meromorphic functions on Riemann surfaces.

Show that a map $f : S \to \mathbb{C}P^1$ is meromorphic if and only if locally f is expressible as a quotient of holomorphic functions (where the denominator is not identically zero).

Show that if f, g are two meromorphic functions on a compact connected Riemann surface having the same zeros and the same poles (including multiplicities) then $f = \text{constant} \cdot g$.

By comparing Taylor series of $\wp, \wp'$ near ramification points, deduce by the previous part (by viewing the two sides of the equation below as meromorphic functions) that:

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

where $e_1 = \wp(\frac{1}{2}\omega_1)$, $e_2 = \wp(\frac{1}{2}\omega_2)$, $e_3 = \wp(\frac{1}{2}(\omega_1 + \omega_2))$, $\infty = \wp(0)$ are the branch points of $\wp$.

Footnote hint

Hints. It helps if you first ask yourself what local holomorphic coordinate you would use at solutions (z, w) of $w^2 = z$ (recall from lecture notes the discussion of the square root $z^{1/2}$). Then try to build the solution set S_1 by gluing two cut-domains: two copies of $\mathbb{C}$ cut from 1 to 2. Just like for $\operatorname{Log} z$ in lectures, each subset you cut gives rise to *two* copies of that subset in the Riemann surface. In order to be able to draw the Riemann surface inside $\mathbb{R}^3$, it is convenient to reflect one of the cut-domains about the x-axis. Near infinity, try using the coordinates $X = \frac{1}{z}$ and $Y = \frac{w}{z}$ instead of z, w, and ask yourself what happens for $X = 0$ (corresponding to "$z = \infty$"). For S_2 you will need a second cut, from 3 to ∞, and try instead $Y = \frac{w}{z^2}$.