ASSIGNMENT I

INSTRUCTION TO STUDENTS: ATTEMPT ALL QUESTIONS

Remark. Reading lecture notes is surprisingly helpful.

Exercise 1. Topological surfaces by gluing the sides of a square.

Starting from the square, how many different topological spaces can you produce, if you are allowed to identify the sides? (make as many identifications as you like, in pairs or more)

Which of these topological spaces do you think are topological surfaces?

You may have heard that it is enough to specify a direction along which you identify the edges. In what precise sense is this true, and why?

Exercise 2. The real projective plane

Recall that $\mathbb{RP}^2$ can be thought of as the collection of all 1-dimensional vector subspaces inside $\mathbb{R}^3$. Explain why a 2-dimensional vector subspace of $\mathbb{R}^3$ determines a curve in $\mathbb{RP}^2$. This is called a *line* in $\mathbb{RP}^2$.

Show that two distinct lines in $\mathbb{RP}^2$ always intersect in exactly one point.

Show that through two points there passes a unique line.

By viewing $\mathbb{RP}^2$ as the 2-sphere modulo the action of the antipodal map $x \mapsto -x$, show that every line in $\mathbb{RP}^2$ is homeomorphic to a circle. Convince yourself that, using the notion of distance on $\mathbb{RP}^2$ inherited from S^2, the shortest path connecting two points of $\mathbb{RP}^2$ is along the unique line through those two points.

Exercise 3. $\mathbb{CP}^1$ as a quotient of spheres.

Recall that the complex projective space $\mathbb{C}P^1$ is the space of complex lines through 0 in $\mathbb{C}^2$. By thinking about how a complex 1-dimensional vector space intersects the sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, show that $\mathbb{C}P^1$ as a topological surface can be viewed as a quotient

$$\mathbb{C}P^1 = S^3/S^1$$

where you need to explain how the group S^1 acts on S^3.

Exercise 4. The Möbius band.

The open Möbius band is the quotient

$$M = [0,1] \times (0,1) \,/\, ((0,y) \sim (1,1-y) \text{ for all } y \in (0,1)).$$

Briefly explain why M is a smooth surface. Find (a homeomorphic copy of) M inside the real projective space $\mathbb{R}P^2$ and inside the Klein bottle K. The Möbius band $\overline{M}$, is obtained by replacing $(0,1)$ by $[0,1]$ above. Show[1] that the boundary of $\overline{M}$ is homeomorphic to S^1. Show that $\mathbb{R}P^2 =$ (closed disc) $\cup\, \overline{M}$ glued along the circular boundary, and $K = \overline{M} \cup \overline{M}$.