STATISTICS AND RANDOM PROCESSES

INTRODUCTION TO STOCHASTIC PROCESSES

In electrical systems voltage or current waveforms are used as signals for collecting, transmitting or processing information, as well as for controlling and providing power to a variety of devices. These signals (voltage or current waveforms) are functions of time and are of two classes—deterministic and random. Deterministic signals can be described by the usual mathematical functions with time t as the independent variable. But a random signal always has some element of uncertainty associated with it and hence it is not possible to determine its value exactly at any given point of time. However, we may be able to describe the random signal in terms of its average properties such as the average power in the random signal, its spectral distribution and the probability that the signal amplitude exceeds a given value. The probabilistic model used for characteristing a random signal is called a *random process* or *stochastic process*.

A random variable (RV) is a rule (or function) that assigns a real number to every outcome of a random experiment, while a random process is a rule (or function) that assigns a time function to every outcome of a random experiment. For example, consider the random experiment of tossing a dice at $t = 0$ and observing the number on the top face. The sample space of this experiment consists of the outcomes $\{1, 2, 3, \cdots, 6\}$. For each outcome of the experiment, let us arbitrarily assign a function of time t $(0 \le t < \infty)$ in the following manner.

Outcome:	1	2	3	4	5	6
Function of time:	$x_1(t) = -4$	$x_2(t) = -2$	$x_3(t) = 2$	$x_4(t) = 4$	$x_5(t) = -t/2$	$x_6(t) = t/2$

The set of functions $\{x_1(t), x_2(t), \cdots, x_6(t)\}$ represents a random process.

Definition: A random process is a collection (or ensemble) of RVs $\{X(s, t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set).

The set of possible values of any individual member of the random process is called *state space*. Any individual member itself is called a *sample function* or a realisation of the process.

Note:

(i) If s and t are fixed, $\{X(s, t)\}$ is a number.

(ii) If t is fixed, $\{X(s, t)\}$ is a RV.

(iii) If s is fixed, $\{X(s, t)\}$ is a single time function.

(iv) If s and t are variables, $\{X(s , t)\}$ is a collection of RVs that are time functions.

If the parameter set T is discrete, the random process will be denoted by $\{X(n)\}$ or $\{X_n\}$

Classification of Stochastic Processes

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

(i) If both T and S are discrete, the random process is called a *discrete random sequence*. For example, if X_n represents the outcome of the nth toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \cdots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

(ii) If T is discrete and S is continuous, the random process is called a *continuous random sequence*.
For example, if X_n represents the temperature at the end of the nth hour of a day, then $\{X_n, 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

(iii) If T is continuous and S is discrete, the random process is called a *discrete random process*.
For example, if $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete random process, since $S = \{0, 1, 2, 3, \cdots\}$.

(iv) If both T and S are continuous, the random process is called a *continuous Random process*. For example, if $X(t)$ represents the maximum temperature at a place in the interval $(0, t)$, $\{X(t)\}$ is a continuous random process. In the names given above, the word 'discrete' or 'continuous' is used to refer to the nature of S and the word 'sequence' or 'process' is used to refer to the nature of T.

Special Classes of Random Processes

The important feature of a random process is the relationship among the members of the family. Usually the nature of relationship is understood by the joint distribution function of the member RVs. A random process is said to be specified only when the parameter set, the state space and the nature of dependence relationship existing among the members of the family are specified.

Based on the dependence relationship among the members of the process, random processes are classified broadly into a few special types such as the ones explained below.

(i) Markov process

If, for $t_1 < t_2 < t_3 < \cdots < t_n < t$, $P\{X(t) \le x/X(t_1) = x_1, X(t_2) = x_2, \cdots, X(t_n) = x_n\} = P\{X(t) \le x/X(t_n) = x_n\}$, then the process $\{X(t)\}$ is called a Markov process.

In other words, if the future behaviour of a process depends only on the present state, but not on the past, the process is a Markov process.

A discrete parameter Markov process is called a Markov chain.

(ii) Process with independent increments

If, for all choices of $t_1, t_2, \cdots, t_n$ such that $t_1 < t_2 < t_3 < \cdots < t_n$, the random variables $X(t_2) - X(t_1), X(t_3) - X(t_2), \cdots, X(t_n) - X(t_{n-1})$ are independent, then the process $\{X(t)\}$ is said to be a random process with independent increments.

If $T = \{0, 1, 2, \cdots\}$ is the parameter set for $\{X_n\}$, then $\{Z_n\}$, where $Z_0 = X_0$ and $Z_n = X_n - X_{n-1}$, is a random sequence with independent increments if the RVs $Z_0, Z_1, Z_2, \cdots$, are independent.

(iii) Stationary processes

If certain probability distribution or averages do not depend on t, then the random process $\{X(t)\}$ is called stationary.

RANDOM WALK

Definition:

Suppose we toss a fair coin every T seconds and instantly after each toss we move a distance d to the right if heads show and to the left if tails show. If the process starts at $t = 0$, our position at time $t = nT$ is a random sequence $X(nT)$ that may be simply denoted as $X(n)$ also. The process $\{X(nT)\}$ is called a random walk.

Markov Process

Another interesting model of a random process is the one in which the value of the random process depends only upon the most recent previous value and is independent of all values in the more distant past. Such a

model is called a Markov model. Hence Markov process is one in which the future value is independent of the past values, given the present value.

Consider the experiment of tossing a fair coin a number of times. The possible outcomes at each trial are two – 'head' with probability ½ and 'tail' with probability ½. If we denote the outcomes of the nth toss, which is a RV, by X_n and the outcomes 'head' and 'tail' by 1 and 0 respectively, then

$P\{X_n = 1\} = 1/2$ and $P\{X_n = 0\} = 1/2$; $n = 1, 2, ...$

Thus we have a sequence of independent RVs $X_1, X_2, \ldots$ since the trials are independent and hence the outcome of the nth trial does not depend in any way on the previous trials.

Consider now the RV that represents the total number of heads in the first n trials and is given by $S_n = X_1 + \ldots + X_n$. The possible values of S_n are 0, 1, 2, …, n. If S_n = k (k=0, 1, 2, … ,n), then the RV S_{n+1} (= $S_n + X_{n+1}$) can assume only 2 possible values, manely k+1 [if the $n+1$th trial results in a head] and k [if the $n+1$th trial results in a tail].

Thus $$P\{S_{n+1} = k + 1/S_n = k\} = \frac{1}{2}$$

$$P\{S_{n+1} = k/S_n = k\} = \frac{1}{2}$$

These probabilities are not at all affected by the values of the RVs $S_1, S_2, \ldots, S_{n-1}$. Also the conditional probability of S_{n+1} given S_n depends on the value of S_n and not on the manner in which the value of S_n was reached. This is a simple example of a Markov chain. Random processes $\{X(t)\}$ (with Markov property) which take discrete values, whether t is discrete or continuous, are called ***Markov Chains***. Poisson process is a continuous time Markov chain. In this section we discuss discrete time Markov chain.

Definition of a Markov Chain

If, for all n, $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}\}$, then the process $\{X_n\}$, $n = 0, 1, \ldots$, is called a Markov chain.

$(a_1, a_2, \ldots, a_n, \ldots)$ are called the states of the Markov chain. The conditional probability $P\{X_n = a_j/X_{n-1} = a_i\}$ is called the ***one-step transition probability*** from state a_i to state a_j at the nth step (trial) and is denoted by $p_{ij}(n-1, n)$.

If the one-step transition probability does not depend on the step, i.e., $p_{ij}(n-1, n) = p_{ij}(m-1, m)$ the Markov chain is called a ***homogeneous Markov chain*** or the chain is said to have stationary transition probabilities. The use of the word 'stationary' does not imply a stationary random sequence.

When the Markov chain is homogeneous, the one-step transition probability is denoted by p_{ij}. The matrix $P = \{p_{ij}\}$ is called (one-step) ***transition probability matrix***, shortly, tpm.

Note *The tpm of a Markov chain is a stochastic matrix, since* $p_{ij} \geq 0$ *and* $\sum_j p_{ij} = 1$ *for all i, i.e., the sum of all the elements of any row of the tpm is 1. This is obvious because the transition from state* a_i *to any one of the states (including* a_i *itself) is a certain event.*

The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, i.e., $P\{X_n = a_j/X_0 = a_i\}$ is called the ***n-step transition probability*** and denoted by $p_{ij}(n)$.

Note $p_{ij}^{(1)} = p_{ij}$.

Let us consider an example in which we explain how the tpm is formed for a Markov chain. Assume that a man is at an integral point of the x-axis between the origin and the point $x = 3$. He takes a unit step either to the right with probability 0.7 or to the left with probability 0.3, unless he is at the origin when he takes a step to the right to reach $x = 1$ or he is at the point $x = 3$, when he takes a step to the left to reach $x = 2$. The chain is called 'Random walk with reflecting barriers'.

The tpm is given below:

$$\begin{array}{c} \\ \text{States of } X_{n-1} \end{array} \begin{array}{c} \begin{array}{cccc} \text{States of } X_n & & & \\ 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

Note *p_{23} = the element in the 2nd row, 3rd column of this tpm = 0.7. This means that, if the process is at state 2 at step (n – 1), the probability that it moves to state 3 at step n = 0.7, where n is any positive integer.*

Definition: If the probability that the process is in state a_i is p_i ($i = 1, 2, \ldots, k$) at any arbitrary step, then the row vector $p = (p_1, p_2, \ldots, p_k)$ is called ***the probability distribution of the process*** at that time. In particular, $p^{(0)} = \{p_1^{(0)}, p_2^{(0)}, \ldots, p_k^{(0)}\}$ is the initial probability distribution.

[**Remark:** The transition probability matrix together with the initial probability distribution completely specifies a Markov chain $\{X_n\}$. In the example given above, let us assume that the initial probability distribution of the chain is

$$p^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

i.e., $P\{X_0 = i\} = 1/4,\ i = 0, 1, 2, 3$

Then we have, for the example given above,

$$P\{X_1 = 2/X_0 = 1\} = 0.7; \quad P\{X_2 = 1/X_1 = 2\} = 0.3,$$

$$P\{X_2 = 1, X_1 = 2/X_0 = 1\}$$

$$= P\{X_2 = 1/X_1 = 2\} \times P\{X_1 = 2/X_0 = 1\}$$

$$= 0.3 \times 0.7 = 0.21 \qquad (1)$$

$$P\{X_2 = 1, X_1 = 2, X_0 = 1\}$$

$$= P\{X_0 = 1\} \times P\{X_2 = 1, X_1 = 2/X_0 = 1\}$$

$$= 1/4 \times 0.21 = 0.0525 \quad \text{[by (1)]} \qquad (2)$$

$$P\{X_3 = 3, X_2 = 1, X_1 = 2, X_0 = 1\}$$

$$= P\{X_2 = 1, X_1 = 2, X_0 = 1\}$$

$$\times P\{X_3 = 3/X_2 = 1, X_1 = 2, X_0 = 1\}$$

$$= 0.0525\, P\{X_3 = 3/X_2 = 1\}$$

(Markov property) [by (2)]

$$= 0.0525 \times 0 = 0$$

Chapman - Kolmogorov Theorem

If P is the tpm of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n.

i.e., $= [p_{ij}^{(n)}] = [p_{ij}]^n$

Proof $P_{ij}^{(2)} = P\{X_2 = j/X_0 = i\}$, since the chain is homogeneous.

The state j can be reached from the state i in 2 steps through some intermediate state k.

Now $p_{ij}^{(2)} = P\{X_2 = j/X_0 = i\} = P\{X_2 = j, X_1 = k/X_0 = i\}$

$= P\{X_2 = j/X_1 = k, X_0 = i\}\, P\{X_1 = k/X_0 = i\}$

$= p_{kj}^{(1)}\, p_{ik}^{(1)}$

$= p_{ik}\, p_{kj}$

Since the transition from state i to state j in 2 steps can take place through any one of the intermediate states, k can assume the values 1, 2, 3, ... The transitions through various intermediate states are mutually exclusive.

Hence $p_{ij}^{(2)} = \sum_k p_{ik}\, p_{kj}$

i.e., the ij-th element of 2 step tpm =
the ij-th element of the product of the 2 one-step tpm's

i.e., $P^{(2)} = P^2$

Now $p_{ij}^{(3)} = P\{X_3 = j/X_0 = i\}$

$= \sum_k P\{X_3 = j/X_2 = k\}\, P\{X_2 = k/X_0 = i\}$

$= \sum_k p_{kj}\, p_{ik}^{(2)}$

$= \sum_k p_{ik}^{(2)}\, p_{kj}$

Similarly $p_{ij}^{(3)} = \sum_k p_{ik}\, p_{kj}^{(2)}$

i.e., $P^{(3)} = P^2\, P = P\, P^2 = P^3$

Proceeding further in a similar way, we get

$$P^{(n)} = P^n$$

For example, consider the problem of Random walk with reflecting barriers, discussed above, for which the tpm is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.51 & 0 & 0.49 \\ 0.09 & 0 & 0.91 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{pmatrix}$$

From this matrix, we see that $p_{11}^{(2)} = 0.51$. This is so, because

$$p_{11}^{(2)} = p_{10}\, p_{01} + p_{11} P_{11} + p_{12}\, p_{21} + p_{13}\, p_{31}$$
$$= (0.3)\,(1) + (0)\,(0) + (0.7)\,(0.3) + (0)\,(0) = 0.51$$

Definition: A stochastic matrix P is said to be a ***regular matrix***, if all the entries of P^m (for some positive integer m) are positive. A homogeneous Markov chain is said to be ***regular*** if its tpm is regular.

We state below two theorems without proof:

1. If $p = \{p_i\}$ is the state probability distribution of the process at an arbitrary time, then that after one step is pP, where P is the tpm of the chain and that after n steps in pP^n.
2. If a homogeneous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called the ***stationary (state) distribution*** or ***steady-state distribution*** of the Markov chain.

   That is, $\lim_{n \to \infty} \{p^{(n)}\} = \pi$, where the state probability distribution at step n, $p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_k^{(n)})$ and the stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ are row vectors.
3. Moreover, if P is the tpm of the regular chain, then $\pi P = \pi$ (π is a row

vector)

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