STATISTICS AND RANDOM PROCESSES

PART II

Classification of States of a Markov Chain

If $P_{ij}^{(n)} > 0$ for some n for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be irreducible. The TPM of irreducible Markov chain is an irreducible matrix. Otherwise, the chain is said to be reducible.

State i of a Markov chain is called a return state if $P_{ii}^{(n)} > 0$ for some $n \geq 1$.

The period d_i of a return state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$, i.e., $d_i = \text{GCD}\{m: p_{ii}^{(m)} > 0\}$. State i is said to be ***periodic*** with period d_i if $d_i > 1$ and ***aperiodic*** if $d_i = 1$.

Obviously state i is aperiodic if $p_{ii} \neq 0$. The probability that the chain returns to state i, having started from state i, for the first time at the nth step (or after n transitions) is denoted by $f_{ii}^{(n)}$ and *called the **first return time probability*** or ***the recurrence time probability***. $\{n, f_{ii}^{(n)}\}$, $n = 1, 2, 3, \ldots$, is the distribution of recurrence times of the state i.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called *the **mean recurrence time*** of the state i.

A state i is said to be ***persistent*** or ***recurrent*** if the return to state i is certain, i.e., if $F_{ii} = 1$. The state i is said to be ***transient*** if the return to state i is uncertain, i.e., if $F_{ii} < 1$. The state i is said to be ***nonnull persistent*** if its mean recurrence time μ_{ii} is finite and ***null persistent***, if $\mu_{ii} = \infty$.

A nonnull persistent and aperiodic state is called ***ergodic***.

We give below two theorems, without proof, which will be helpful to classify the states of a Markov chain.

1. If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all nonnull persistent. All its states are either aperiodic or periodic with the same period.
2. If a Markov chain is finite irreducible, all its states are nonnull persistent.

Example 1 *The transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, 3, \ldots$, having 3 states 1, 2 and 3 is*

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

and the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$.
Find (i) $P\{X_2 = 3\}$ and (ii) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$.

Solution

$$P^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(i) $P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3/X_0 = i\} \times P\{X_0 = i\}$

$= p_{13}^{(2)} P\{(X_0 = 1) + p_{23}^{(2)} P(X_0 = 2) + p_{33}^{(2)} P(X_0 = 3)$

$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$

$= 0.182 + 0.068 + 0.029$

$= 0.279$

(ii) $P\{X_1 = 3/X_0 = 2\} = p_{23} = 0.2$ (1)

$P\{X_1 = 3, X_0 = 2\} = P\{X_1 = 3/X_0 = 2\} \times P\{X_0 = 2\}$

$= 0.2 \times 0.2 = 0.04$ [by (1)] (2)

$P\{X_2 = 3, X_1 = 3, X_0 = 2\} = P\{X_2 = 3/X_1 = 3, X_0 = 2\} \times P\{X_1 = 3, X_0 = 2\}$

$= P\{X_2 = 3/X_1 = 3\} \times P\{X_1 = 3, X_0 = 2\}$

(by Markov property)

$= 0.3 \times 0.04$ [by (2)]

$= 0.012$ (3)

$P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$

$= P\{X_3 = 2/X_2 = 3, X_1 = 3, X_0 = 2\}$

$\times\ P\{X_2 = 3, X_1 = 3, X_0 = 2\}$

$= P\{X_3 = 2/X_2 = 3\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$

(by Markov property)

$= 0.4 \times 0.012$ [by (3)]

$= 0.0048$

Example 2 *A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occuring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$.*

Find also P^2 and $P(X_2 = 6)$

Solution State space: {1, 2, 3, 4, 5, 6}

The tpm is formed using the following analysis.

Let X_n = the maximum of the numbers occurring in the first n trials = 3, say

Then X_{n+1} = 3, if the $(n + 1)$th trial results in 1, 2 or 3

= 4, if the $(n + 1)$th trial results in 4

= 5, if the $(n + 1)$th trial results in 5

= 6, if the $(n + 1)$th trial results in 6

$$\therefore \quad P\{X_{n+1} = 3/X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$P\{X_{n+1} = i/X_n = 3\} = \frac{1}{6}, \text{ when } i = 4, 5, 6$$

Therefore, the transition probability matrix of the chain is

$$P = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^2 = \frac{1}{36}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

Initial state probability distribution is $p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ since all the values 1, 2, …, 6 are equally likely.

$$P\{X_2 = 6\} = \sum_{i=1}^{6} P\{X_2 = 6/X_0 = i\} \times P\{X_0 = i\}$$

$$= \frac{1}{6}\sum_{i=1}^{6} p_{i6}^{(2)}$$

$$= \frac{1}{6} \times \frac{1}{36} \times (11 + 11 + 11 + 11 + 11 + 36)$$

$$= \frac{91}{216}$$

Example 3 *A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a 6 appeared. Find (i) the probability that he takes a train on the third day and (ii) the probability that he drives to work in the long run.*

Solution The travel pattern is a Markov chain, with state space = (train, car)

The tpm of the chain is

$$P = \begin{array}{c} \\ T \\ C \end{array}\begin{array}{c} \begin{matrix} T & C \end{matrix} \\ \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

The initial state probability distribution is $p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$,

since P(travelling by car) = P(getting 6 in the toss of the dice)

$$= \frac{1}{6}$$

and P(travelling by train) = $\frac{5}{6}$

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on the third day) = $\frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π, $\pi P = \pi$

i.e., $$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

i.e., $$\frac{1}{2}\pi_2 = \pi_1 \qquad (1)$$

and $$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \qquad (2)$$

Equations (1) and (2) are one and the same.

Therefore, consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

Solving, $\pi_1 = \frac{1}{3}$ and $\pi_2 = \frac{2}{3}$

$\therefore$ P\{the man travels by car in the long run\} = $\frac{2}{3}$.

Example 4 *Consider a communication system which transmits the 2 digits 0 and 1 through several stages. Let X_n ($n \geq 1$) be the digit leaving the nth stage of the system and X_0 be the digit entering the first stage (or leaving the 0th stage). At each stage there is a constant probability q that the digit which enters will be transmitted unchanged (i.e., the digit will remain unchanged when it leaves) and the probability p otherwise (i.e., the digit changes when it leaves), where $p + q = 1$. Write down the tpm P of the homogeneous two-state Markov chain $\{X_n\}$. Find P^m, P^∞ and the conditional probability that the digit entering the first stage is 0, given that the digit leaving the mth stage is 0. Assume that the initial state probability distribution is $p^{(0)} = (a, 1 - a)$.*

Solution State space = (0, 1); $P \equiv$ State of X_n

$$\begin{array}{cc} & \text{State of } X_{n+1} \\ & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} q & p \\ p & q \end{pmatrix} \end{array}$$

Now

$$P^2 = \begin{pmatrix} q & p \\ p & q \end{pmatrix}\begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

$$= \begin{pmatrix} p^2 + q^2 & 2pq \\ 2pq & p^2 + q^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\left[(q+p)^2 + (q-p)^2\right] & \frac{1}{2}\left[(q+p)^2 - (q-p)^2\right] \\ \frac{1}{2}\left[(q+p)^2 - (q-p)^2\right] & \frac{1}{2}\left[(q+p)^2 + (q-p)^2\right] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^2 & \frac{1}{2} - \frac{1}{2}r^2 \\ \frac{1}{2} - \frac{1}{2}r^2 & \frac{1}{2} + \frac{1}{2}r^2 \end{pmatrix}, \text{ where } q - p = r$$

$$P^3 = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^3 & \frac{1}{2} - \frac{1}{2}r^3 \\ \frac{1}{2} - \frac{1}{2}r^3 & \frac{1}{2} + \frac{1}{2}r^3 \end{pmatrix}$$

The values of P^2 and P^3 make us guess that

$$P^m = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^m & \frac{1}{2} - \frac{1}{2}r^m \\ \frac{1}{2} - \frac{1}{2}r^m & \frac{1}{2} + \frac{1}{2}r^m \end{pmatrix}$$

It is correct as can be proved by induction as follows:

$$P^{m+1} = \begin{pmatrix} q & p \\ p & q \end{pmatrix}\begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^m & \frac{1}{2} - \frac{1}{2}r^m \\ \frac{1}{2} - \frac{1}{2}r^m & \frac{1}{2} + \frac{1}{2}r^m \end{pmatrix}$$

$$= \begin{pmatrix} \frac{q}{2} + \frac{q}{2}r^m + \frac{p}{2} - \frac{p}{2}r^m & \frac{q}{2} - \frac{q}{2}r^m + \frac{p}{2} + \frac{p}{2}r^m \\ \frac{p}{2} + \frac{p}{2}r^m + \frac{q}{2} - \frac{q}{2}r^m & \frac{p}{2} - \frac{p}{2}r^m + \frac{q}{2} + \frac{q}{2}r^m \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^{m+1} & \frac{1}{2} - \frac{1}{2}r^{m+1} \\ \frac{1}{2} - \frac{1}{2}r^{m+1} & \frac{1}{2} + \frac{1}{2}r^{m+1} \end{pmatrix}$$

$$p^{\infty} = \lim_{m \to \infty} \left(P^m\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{since } |r| < 1$$

Now $\quad P\{X_m = 0, X_0 = 0\} = P\{X_m = 0/X_0 = 0\} \times P\{X_0 = 0)$

$= ap_{00}^{(m)}$

and $\quad P\{X_m = 0, X_0 = 1\} = bp_{10}^{(m)}\ b = 1 - a$

Now $\quad P\{X_0 = 0/X_m = 0\} = \dfrac{p\{X_0 = 0\} \times P\{X_m = 0/X_0 = 0\}}{p\{X_0 = 0\} \times p_{00}^{(m)} + p\{X_0 = 1\} \times P_{10}^{(m)}}$

(by Baye's theorem)

$$= \frac{a\left\{\frac{1}{2} + \frac{1}{2} r^m\right\}}{a\left\{\frac{1}{2} + \frac{1}{2} r^m\right\} + b\left\{\frac{1}{2} - \frac{1}{2} r^m\right\}}$$

$$= \frac{a\left(1 + r^m\right)}{1 + (a - b) r^m}, \text{ where } b = 1 - a$$

Example 5 *A gambler has Rs 2/-. He bets Re 1 at a time and wins Re 1 with probability 1/2. He stops playing if he loses Rs 2 or wins Rs 4 (a) What is the tpm of the related Markov chain? (b) What is the probability that he has lost his money at the end of 5 plays? (c) What is the probability that the game lasts more than 7 plays?*

Solution Let X_n represent the amount with the player at the end of the nth round of the play.

State space of $\{X_n\} = (0, 1, 2, 3, 4, 5, 6)$, as the game ends, if the player loses all the money ($X_n = 0$) or wins Rs. 4, i.e., has Rs 6 ($X_n = 6$). The tpm of the Markov chain is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note *This is called a random walk with* ***absorbing barriers*** *at 0 and 6, since the chain cannot come out of the states 0 and 6, once it has entered them.*

The initial probability distribution of $\{X_n\}$ is $p(0) = (0, 0, 1, 0, 0, 0, 0)$, as the player has got Rs 2/- to start with.

$p^{(1)} = p^{(0)} P = (0, 1/2, 0, 1/2. 0, 0, 0)$

$p^{(2)} = p^{(1)} P = (1/4, 0, 1/2, 0, 1/4, 0, 0)$

$p^{(3)} = p^{(2)} P = (1/4, 1/4, 0, 3/8, 0, 1/8, 0)$

$p^{(4)} = p^{(3)} P = (3/8, 0, 5/16, 0, 1/4, 0, 1/16)$

$p^{(5)} = p^{(4)} P = (3/8, 5/32, 0, 9/32, 0, 1/8, 1/16)$

$P\{$the man has lost his money at the end of 5 plays$\}$

$= P\{X_5 = 0\} =$ the entry corresponding to state 0 in $p^{(5)}$

$= 3/8$

Again $p^{(6)} = p^{(5)} P = (29/64, 0, 7/32, 0, 13/64, 0, 1/8)$
$p^{(7)} = p^{(6)} P = (29/64, 7/64, 0, 27/128, 0, 13/128, 1/8)$
P {the game lasts more than 7 rounds} = P {the system is neither in state 0 nor in 6 at the end of the Seventh round}
$= P\{X_7 = 1, 2, 3, 4 \text{ or } 5\}$
$= 7/64 + 0 + 27/128 + 0 + 13/128 = 27/64.$

Example 6 There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are interchanged. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?

State space of the chain $\{X_n\} = (0, 1, 2)$, since the number of balls in the urn A is always 2.

Let the tpm of the chain $\{X_n\}$ be

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix} \end{array}$$

$p_{00} = 0$ (since the state cannot remain at 0 after interchange of marbles)

$p_{02} = p_{20} = 0$ (since the number of red marbles in urn cannot increase or decrease by 2 in one interchange)

To start with, A contains 0 red marble. After an interchange, A will contain 1 red marble (and 1 white marble) certainly.

$\therefore \quad p_{01} = 1.$

Let $X_n = 1$, i.e., A contains 1 red marble (and 1 white marble) and B contains 1 white and 2 red marbles.

Then $X_{n+1} = 0$, if A contains 0 red marble (and 2 white marbles) and B contains 3 red marbles, i.e., if 1 red marble is chosen from A and 1 white marble is chosen from B and interchanged.

$\therefore\ P\{X_{n+1} = 0 / X_n = 1\} = p_{10} = 1/2 \times 1/3 = 1/6$

Similarly, we can find $p_{12} = 1/2 \times 2/3 = 1/3$

Since P is a stochastic matrix, $p_{10} + p_{11} + p_{12} = 1$

$$\therefore \qquad p_{11} = \frac{1}{2}$$

Similarly, $p_{21} = \frac{2}{3}$ and $p_{22} = 1 - (p_{20} + p_{21}) = \frac{1}{3}$

$$\therefore \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

Now $\quad p^{(0)} = (1, 0, 0)$, as there is no red marble in A in the beginning.

$$p^{(1)} = p^{(0)} P = (0, 1, 0)$$

$$p^{(2)} = p^{(1)} P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$$

$$p^{(3)} = p^{(1)} P = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18}\right)$$

$\therefore \quad P$\{there are 2 red marbles in A after 3 steps\}

$$= P\{X_3 = 2\} = p_2^{(3)} = \frac{5}{18}$$

Let the stationary probability distribution of the chain be $\pi = (\pi_0, \pi_1, \pi_2)$.
By the property of π, $\pi P = \pi$ and $\pi_0 + \pi_1 + \pi_2 = 1$

$$\text{i.e., } (\pi_0, \pi_1, \pi_2)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (\pi_0, \pi_1, \pi_2)$$

$$\text{i.e., } \qquad \frac{1}{6}\pi_1 = \pi_0$$

$$\pi_0 + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 = \pi_1$$

$$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_2$$

and $\qquad \pi_1 + \pi_2 + \pi_3 = 1$

$$\text{Solving, } \qquad \pi_0 = \frac{1}{10}, \pi_1 = \frac{6}{10}, \pi_2 = \frac{3}{10}$$

$\therefore P$\{there are 2 red marbles in A in the long run\} = 0.3.

Example 7 Find the nature of the states of the Markov chain with the tpm

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Solution $P^2 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$; $P^3 = P$

$\therefore \quad P^4 = P^2$

and so on. In general, $P^{2n} = P^2, P^{2n+1} = P$

We note that $p_{00}^{(2)} > 0, p_{01}^{(1)} > 0, p_{02}^{(2)} > 0$

$p_{10}^{(1)} > 0, p_{11}^{(2)} > 0, p_{12}^{(1)} > 0$

$p_{20}^{(2)} > 0, p_{21}^{(1)} > 0, p_{22}^{(2)} > 0$

Therefore, the Markov chain is irreducible.

Also $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \ldots > 0$, for all i, all the states of the chain are periodic, with period 2.

Since the chain is finite and irreducible, all its states are nonnull persistent. All states are not ergodic.

Example 8 *Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.* (MKU — Nov. 96)

Solution The transition probability matrix of the process $\{X_n\}$ is given below:

$$\begin{array}{ll} & \text{State of } X_n \\ \text{State of } X_{n-1} & \quad A \quad B \quad C \end{array}$$

$$p \equiv \begin{matrix} A \\ B \\ C \end{matrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

States of X_n depend only on states of X_{n-1}, but not on states of $X_{n-2}, X_{n-3}, \ldots$, or earlier states. Therefore, $\{X_n\}$ is a Markov chain.

Now $P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$; $P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

$p_{11}^{(3)} > 0, p_{13}^{(2)} > 0, p_{21}^{(2)} > 0, P_{22}^{(2)} > 0, p_{33}^{(2)} > 0$ and all other $p_{ij}^{(1)} > 0$. Therefore, the chain is irreducible.

$P^4 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$; $P^5 = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix}$; $P^6 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 3/8 & 1/2 \\ 1/8 & 3/8 & 3/8 \end{pmatrix}$

and so on.

We note that $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(5)}; p_{ii}^{(6)}$ etc are > 0 for $i = 2, 3$, and GCD of 2, 3, 5, 6, … = 1.

Therefore, the states 2 and 3 (i.e., B and C) are periodic with period 1. i.e., aperiodic.

We note that $p_{11}^{(3)}, p_{11}^{(5)}, p_{11}^{(6)}$ etc. are > 0 and GCD of 3, 5, 6, … = 1

Therefore, the state 1 (i.e., state A) is periodic with period 1, i.e., aperiodic.

Since the chain is finite and irreducible, all its states are nonnull persistent. Moreover all the states are ergodic.

Reference

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