Markov chain

Definitions (Markov chain)

1. If , for all n,

   $$P\,(X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots\ldots, X_0 = a_0)$$

   $$= P(\,X_n = a_n / X_{n-1} = a_{n-1})$$

   Then the process (X_n; $n = 0,1,2,\ldots\ldots\ldots$) is called a **Markov chain**

2. $a_1, a_2, \ldots\ldots\ldots, a_n \ldots\ldots$ are called the states of the Markov chain.

3. The conditional probability $P_{ij}(n\text{-}1, n) = P(\,X_n = a_j / X_{n-1} = a_i)$ is called the **one-step transition probability** from state a_i to state a_j in the n^{th} step.

4. If the one step transition probability does not depend on the step, then the Markov chain is called a **homogeneous Markov chain.**

5. If the Markov chain is homogenous, the one step transition probability is denoted by P_{ij} and the matrix $P = (\,p_{ij}\,)$ is called the **one step transition probability matrix (tpm)**

Note

The tpm of a Markov chain is stochastic matrix. i.e.

a) $p_{ij} \geq 0$ for all i,j

b) $\sum_j p_{ij} = 1$ for all i (i.e. sum of the elements of any row is 1)

6. The conditional probability $P_{ij}^{(n)} = P(X_n = a_j / X_0 = a_i)$ is called the ***n* step transition probability.**

7. Let p_i = probability that the process is in state a_i at any step (i= 1,2,.., k). Then the row vector $p = (p_1, p_2, \ldots\ldots p_k)$ is called the probability distribution of the process at that time.

8. $P^{(0)} = (P_1^{(0)}, P_2^{(0)}, \ldots, P_k^{(0)})$ is called the initial probability distribution where $P_1^{(0)} = P[X_0=1]$, $P_2^{(0)} = P[X_0=2]$………… are the probabilities for states 1,2,….

Note

The tpm together with the initial probability distribution completely specifies a Markov chain.

Classification of states of a Markov Chain

Definitions

1. **Irreducible Chain:** If for every i,j, we can find some n such that $p_{ij}^{(n)} > 0$, then every state can be reached from every other state, and the

Markov chain is said to be irreducible. Otherwise the chain is non-irreducible or reducible.

2. **Return State:** State *i* of a Markov chain is called a return state, if $p_{ij}^{(n)} > 0$ for some *n* > 1.

3. **Periodic State:** The period d_i of a return state *i* is the greatest common divisor of all *m* such that $p_{ij}^{(m)} > 0$ i.e. $d_i = GCD\ (m : p_{ij}^{(m)} > 0\ )$. State i is periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$.

4. The probability that the chain returns to state *i,* having started from state *i,* for the first time at the n^{th} step is denoted by $f_{ii}^{(n)}$ and is called the first return time probability.

5. **Recurrent state:** If $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1,$ the return to state *i* is certain and the state *i* is said to be persistent or recurrent. Otherwise , it is said to betransient.

6. **Non-null persistent state:** $\mu_{ii} = \sum_{n=1}^{\infty} nf_{ii}^{(n)}$ is called the mean recurrence time of the state *i*. if μ_{ii} is finite, the state *i* is non-null persistent. If $\mu_{ii} = \infty$ the state *i* is null persistent.

7. **Ergodic state:** A non-null persistent and aperiodic state is called ergodic.

Theorems used to classify states

1. If a Markov Chain is irreducible, all its states are of the same type. They are either all transient or all null persistent or all non-null persistent.

   All the states are either aperiodic or periodic with the same period.

2. If a Markov chain is finite and irreducible, then all its states are non-null persistent.

Solved problem

1. A housewife busy 3 kinds of cereals, A, B and C. She never busy the same cereal in successive weeks. If she buys cereal A, the next week she buys B. However, if she buys B or C, the next she is 3 times as likely to buy A as the other cereal. In the long run, how often does she buy each of the three cereals?

Solution

The transition probability matrix of the process is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Let $\pi = (\pi_1, \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1, \pi_2, \pi_3] \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\frac{3\pi_2}{4} + \frac{3\pi_3}{4} = \pi_1$$

$$\pi_1 + \frac{\pi_3}{4} = \pi_2$$

$$\frac{\pi_2}{4} = \pi_3$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{15}{35}, \quad \pi_2 = \frac{16}{35}, \quad \pi_3 = \frac{4}{35}$$

In the long run, probability of buying $A = \frac{15}{35}$

In the long run, probability of buying $B = \frac{16}{35}$

In the long run, probability of buying $C = \frac{4}{35}$

2. Assume that the weather in a certain locality can be modeled as the homogeneous Markov chain whose transition probability matrix is given below.

	Tomorrow's weather		
Today's weather	Fair	Cloudy	Rainy
Fair	0.8	0.15	0.05
Cloudy	0.5	0.3	0.2
Rainy	0.6	0.3	0.1

If the initial state distribution is given by

$$P^{(0)} = (0.7, 0.2, 0.1)$$

Find $P^{(2)}$ and $\lim_{n\to\infty} P^{(n)}$

Solution

The transition probability matrix is

$$P = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

The probability distribution $P^{(1)}$ is

$$P^{(1)} = P^{(0)}P = [\,0.7\ 0.2\ \ 0.1] \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

$$= [\,0.72\ \ 0.195\ \ 0.085]$$

$$P^{(2)} = P^{(1)}P = [\,0.72\ \ 0.195\ \ 0.085] \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

$$= [\,0.7245\ \ 0.192\ \ 0.0835]$$

Let $\pi = (\pi_1 , \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1 , \pi_2, \pi_3] \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{bmatrix} = (\pi_1 , \pi_2, \pi_3)$$

$$0.8\pi_1 + 0.5\pi_2 + 0.6\pi_3 = \pi_1$$

$$0.15\pi_1 + 0.3\pi_2 + 0.3\pi_3 = \pi_2$$

$$0.05\pi_1 + 0.2\pi_2 + 0.1\pi_3 = \pi_3$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{114}{157}, \quad \pi_2 = \frac{30}{157}, \quad \pi_3 = \frac{13}{157}$$

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} \frac{114}{157} & \frac{30}{157} & \frac{13}{157} \end{bmatrix}$$

3. Let $(X_n : n = 1,2,3,\ldots.)$ be a Markov chain with state space $S = \{0,\ 1,\ 2\}$ and one step transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$$

I. Is the chain ergodic? Explain.

II. Find the invariant probabilities.

Solution

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{bmatrix}$$

1st state '0'

$p_{11}^{(2)} > 0, p_{11}^{(3)} > 0, \ldots\ldots\ldots\ldots\ldots\ldots$

$\text{Period} = \text{GCD}\,(2,3,\ldots) = 1$

State 0 is aperiodic.

2nd state '1'

$p_{22}^{(1)} > 0, p_{22}^{(2)} > 0, p_{22}^{(3)} > 0, \dots\dots\dots\dots\dots$

$\text{Period} = \text{GCD}\,(1,2,3,\dots) = 1$

State 1 is aperiodic.

3rd state '2'

$p_{33}^{(2)} > 0, p_{33}^{(3)} > 0, \dots\dots\dots\dots\dots$

$\text{Period} = \text{GCD}\,(2,3,\dots) = 1$

State 2 is aperiodic.

Now

$$p_{11}^{(2)} > 0, p_{12}^{(1)} > 0, p_{13}^{(2)} > 0$$

$p_{21}^{(1)} > 0, p_{22}^{(1)} > 0, p_{23}^{(1)} > 0$

$p_{31}^{(2)} > 0, p_{32}^{(1)} > 0, p_{33}^{(2)} > 0$

The chain is irreducible.

Also, there are only 3 states, so the chain is finite. i.e. the chain is finite irreducible.

Hence, all the states are non null persistent. Since all the states are aperidic and non null persistent, all the states are ergodic.

To find the invariant probabilities

Let the steady state probability distribution be $\pi = [\pi_0 \quad \pi_1 \quad \pi_2]$

Then $\pi P = \pi$.

$$[\pi_0 \quad \pi_1 \quad \pi_2]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$$

$$\frac{\pi_1}{4} = \pi_0$$

$$\pi_0 + \frac{\pi_1}{2} + \pi_2 \quad = \pi_1$$

$$\frac{\pi_1}{4} = \pi_2$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{2}{3}, \quad \pi_0 = \frac{1}{6}, \quad \pi_2 = \frac{1}{6}$$

$$\pi = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix}$$

4. A raining process is considered as a two state Markov chain. If it rains, it is considered to be in state 0 and if it does not rain, the chain is in state 1. The transition probability of the Markov chain is defined as
$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

   I. Find the probability that it will rain after three days from today.

   II. Find also the unconditional probability that it will rain after three days with the initial probabilities of state 0 and state 1 as 0.4 and 0.6 respectively.

Solution

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

I. If it rains today, then probability distribution for today is $[1 \quad 0]$

Probability distribution after 3 days

$$= [1 \quad 0] \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix} = [0.376 \quad 0.624]$$

Probability that it will rain after 3 days = 0.376

II. Given initial probability distribution as $[0.4 \quad 0.6]$, probability distribution after 3 days

$$= [0.4 \quad 0.6] \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix} = [0.3376 \quad 0.6624]$$

Probability that it will after 3 days = 0.3376

5. There are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are interchanged. Let the state a_i of the system be the number of red marbles in A after *i* changes. What is the probability that there are 2 red marbles in urn A after3 step? I n the long run, what is the probability that there are, 2 red marnles in urn A?

Solution

The number of red marbles in A can be 0, 1or 2.

State space is $(0,1,2)$

$P_{00} =$ Probability there are no red balls in A in i^{th} state and $(i+1)^{th}$ state

= 0 since after an interchange one red ball will definitely come to A.

Let the tpm be $\begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$

By the same argument $P_{01} = 1$

Now $P_{00} + P_{01} + P_{02} = 1$ Therefore $P_{02} = 0$

$P_{10} = P(X_{n+1} = 0/X_n = 1)$

= P(Red is taken from urn A and white is taken from urn B)

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

P_{12}= P(1 white is taken from A and 1 red is taken from B)

$$= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

Now

$$P_{10} + P_{11} + P_{12} = 1$$

$$\frac{1}{6} + P_{11} + \frac{1}{3} = 1$$

$$P_{11} = \frac{1}{2}$$

P_{21}= P(1 red is taken from A and 1 white is taken from B)

$$= 1 \times \frac{2}{3} = \frac{2}{3}$$

Now

$$P_{20} + P_{21} + P_{22} = 1$$

$$P_{22} = \frac{1}{3}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Before the process starts, there is no red ball in A,

Initial distribution is $P^{(0)} = [1 \;\; 0 \;\; 0]$

$$P^{(1)} = P^{(0)}P = [1 \;\; 0 \;\; 0] \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = [0 \;\; 1 \;\; 0]$$

$$P^{(2)} = P^{(1)}P = [0 \;\; 1 \;\; 0] \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \end{bmatrix}$$

P (there are 2 red marbles after 3 steps) $= \frac{5}{18}$

Let the steady state probability distribution be $\pi = [\pi_0 \quad \pi_1 \quad \pi_2]$

Then $\pi P = \pi$.

$$[\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$$

$$\frac{\pi_1}{6} = \pi_0$$

$$\pi_0 + \frac{\pi_1}{2} + \frac{2\pi_2}{3} = \pi_1$$

$$\frac{\pi_1}{3} + \frac{\pi_2}{3} = \pi_2$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{1}{10}, \quad \pi_2 = \frac{6}{10}, \quad \pi_3 = \frac{3}{10}$$

$$\pi = \begin{bmatrix} \frac{1}{10} & \frac{6}{10} & \frac{3}{10} \end{bmatrix}$$

In the long run, probability of 2 red marbles in run A = $\frac{3}{10}$

6. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a 6 appeared. Find

    1. The probability that he takes a train on the third day, and

    2. The probability that he drives to work in the long run.

Solution

The state space of the Markov chain is { Train, Car}. The transition probability matrix is $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Now , on the first day,

P(driving to work) = P(getting 6) $= \frac{1}{6}$

P(taking a train) $= \frac{5}{6}$

The initial probability distribution is

$$p^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Now

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

The probability distribution for the third day

i.e.

$$p^{(3)} = p^{(1)} P^2$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$$

Probability that he takes a train on the third day $= \frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1 \quad \pi_2]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\frac{\pi_2}{2} = \pi_1$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2$$

Also $\pi_1 + \pi_2 = 1$

Solving, we get,

$$\pi_1 = \frac{1}{3}, \qquad \pi_2 = \frac{2}{3}$$

$$\pi = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The probability that the man drives to work in the long run $= \frac{2}{3}$

7. Let $\{X_n; \quad n = 1,2,3,\ldots\}$ be a Markov chain on the space $S = \{1,2,3\}$ with one step transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

1. Sketch the transition diagram.
2. Is the chain irreducible? Explain.
3. Is the chain Ergodic? Explain.

Solution

$$P^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \qquad P^5 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$p_{11}^{(2)} > 0, p_{12}^{(1)} > 0, p_{13}^{(2)} > 0$$

$$p_{21}^{(1)} > 0, p_{22}^{(1)} > 0, p_{23}^{(1)} > 0$$

$$p_{31}^{(2)} > 0, p_{32}^{(1)} > 0, p_{33}^{(2)} > 0$$

The chain is irreducible.

Period of state $1 = \text{GCD}\,(2,3,4,5,\dots) = 1$

Period of state $2 = \text{GCD}\,(2,3,4,5,\dots) = 1$

Period of state $3 = \text{GCD}\,(3,5,\dots) = 1$

All the states are eperidic.

There are only 3 state, hence the chain is finite.

All the states are non-null persistent.

Since all the states are a periodic and non-null persistent, they are ergodic.

Reference

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