# Markov chain CONTINUED

This lecture begins with problems and solutions from lecture eight where using Markov's property, we were solving these problems and presenting solutions thereafter. These problems come with solutions to help you grasp the concepts.

**8.** The one step T.P.M of a Markov chain $\{X_n; \quad n = 1,2,3,\ldots\}$ having state space $S = \{1,2,3\}$ is $\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution of $\pi_0\ (0.7, 0.2, 0.1)$ Find

1. $P(X_2 = 3/X_0 = 1)$
2. $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$
3. $P(X_2 = 3)$

**Solution**

$$P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$P(X_2 = 3/X_0 = 1) = \text{elementin } (1,3)^{th} \text{ position in } P^2 = 0.26$$

$$P(X_1 = 3/X_0 = 2) = \text{elementin } (2,3)^{rd} \text{ position in } P = 0.2 \qquad (1)$$

$$P(X_1 = 3, X_0 = 2) = P(X_1 = 3/X_0 = 2)P(X_0 = 2)$$

(Since $A \cap B = P\left(\frac{A}{B}\right)P(B)$)

$= 0.2 \times 0.2$ using (1) and $\pi_0$

$= 0.04$ (2)

$$P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$$

$$= P(X_3 = 3/X_1 = 3, X_0 = 2)P(X_1 = 3, X_0 = 2)$$

$$= P(X_2 = 3/X_1 = 3)P(X_1 = 3, X_0 = 2)$$

Using Markov property

$= (2,3)^{rd}$ element in P $\times 0.04$

$$= 0.3 \times 0.04 = 0.012 \qquad (3)$$

$$P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$$

$$= P(X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2) P(X_3 = 2, X_1 = 3, X_0 = 2)$$

$$= P(X_3 = 2 / X_2 = 3) \times 0.012 \text{ using Markov property and (2)}$$

$$= (3,2)^{\text{th}} \text{ element in } P \times 0.012$$

$$= 0.4 \times 0.012 = 0.0048$$

Now $\pi_2 = \pi_0 \, P^2$

$$= [0.7 \quad 0.2 \quad 0.1] \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$= [0.385 \quad 0.336 \quad 0.279]$$

$P(X_2 = 3) = 0.279$

9. Three boys A,B,C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

**Solution**

The state space is [A B C]. The tpm is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

The state in the $(n+1)^{th}$ step depends only on the $n^{th}$ step and not on the previous steps, Hence the process is Markovian.

Now

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Classification of states

1st state A

$$p_{11}^{(3)} > 0, p_{11}^{(5)} > 0 \dots\dots\dots\dots$$

$$\text{Period} = \text{GCD}\,(3,5,\dots) = 1$$

2nd state B and 3rd state C

$$p_{22}^{(2)} > 0, p_{22}^{(3)} > 0, p_{22}^{(4)} > 0, \dots\dots\dots$$

$$p_{33}^{(2)} > 0, p_{33}^{(3)} > 0, p_{33}^{(4)} > 0, \dots\dots\dots$$

$$\text{Period of B} = \text{GCD}\,(2,3,4,\dots) = 1$$

$$\text{Period of C} = \text{GCD}\,(2,3,4,\dots) = 1$$

All the states A,B,C have period 1. i.e. they are aperidic

Now

$$p_{11}^{(3)} > 0, p_{12}^{(1)} > 0, p_{13}^{(2)} > 0$$

$$p_{21}^{(2)} > 0, p_{22}^{(2)} > 0, p_{23}^{(1)} > 0$$

$$p_{31}^{(1)} > 0, p_{32}^{(1)} > 0, p_{33}^{(2)} > 0$$

The chain is irreducible. Also since there are only 3 states, the chain is finite. i.e. the chain is finite and irreducible.

All the states are non-null persistent. Since all the states are aperiodic and non-null persistent, they are ergodic.

**10.**Consider a Markov chain on (0,1,2) having transition matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

1. Show that the chain is irreducible
2. Find the period
3. Find the stationary distribution

**Solution**

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & \frac{1}{8} & \frac{1}{2} \end{bmatrix}$$

1. $p_{11}^{(2)} > 0, p_{12}^{(2)} > 0, p_{13}^{(1)} > 0$

$p_{21}^{(1)} > 0, p_{22}^{(3)} > 0, p_{23}^{(2)} > 0$

$p_{31}^{(1)} > 0, p_{32}^{(1)} > 0, p_{33}^{(2)} > 0$

The chain is irreducible.

2.

State 0

$p_{11}^{(2)} > 0,\ p_{11}^{(3)} > 0,\ p_{11}^{(4)} > 0, \dots\dots\dots$

$\text{Period } = \text{GCD}\,(2,3,4,\dots) = 1$

State 1

$p_{22}^{(3)} > 0,\ p_{22}^{(5)} > 0,, \dots\dots\dots$

$\text{Period } = \text{GCD}\,(3,5,\dots) = 1$

State 1

$p_{33}^{(2)} > 0,\ p_{33}^{(3)} > 0,, \dots \dots \dots$

$\text{Period} = \text{GCD}\,(2,3,\dots) = 1$

3.

Let the steady state probability distribution be $\pi = [\pi_1 \quad \pi_2 \quad \pi_2]$

Then $\pi P = \pi$.

$$[\pi_1 \quad \pi_2 \quad \pi_3\,]\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = [\pi_1 \quad \pi_2 \quad \pi_3\,]$$

$$\pi_2 \quad + \frac{\pi_3}{3} = \pi_1$$

$$\frac{\pi_3}{2} \ = \pi_2$$

$$\pi_1 = \pi_3$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{2}{5}\ , \qquad \pi_2 = \frac{1}{5},\quad \pi_3 = \frac{2}{5}$$

$$\pi = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

**11.**The tpm of a Markov chain with three states 0, 1, 2 is

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

and the initial state distribution of the chain is $P[X_0 = i] = \frac{1}{3}, i = 0,1,2, \dots \dots$

Find

1. $P[X_2] = 2$
2. $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$
3. $P[\ X_2 = 1, X_0 = 0]$

**Solution**

Given

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$P^{(2)} = P^2 = \text{state of } X_{n-1} = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{bmatrix}$$

From the definition of conditional probability

1.

$P[X_2=2] = \sum_{i=0}^{2} P[X_2 = 2/X_0 = i]P[X_0 = i]$

$= P[X_2 = 2/X_0 = 0]P[X_0 = 0] + P[X_2 = 2/X_0 = 1]P[X_0 = 1]$

$+ P[X_2 = 2/X_0 = 2]P[X_0 = 2]$

$$P^2 = (X_{n-1}) \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{8}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{4}{16} \end{bmatrix}$$

$$P[X_2=2] = P_{02}^{(2)} P[X_0 = 0] + P_{12}^{(2)} P[X_0 = 1] + P_{22}^{(2)} P[X_0 = 2]$$

$$= \frac{1}{3}\left[\frac{1}{16} + \frac{3}{16} + \frac{4}{16}\right] = \frac{1}{6}$$

2.

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$$
$$= P[X_3 = 1/X_2 = 2]P[X_2 = 2/X_1 = 1]P[X_1 = 1/X_0 = 2]P[X_0 = 2]$$

$$= P_{21}^{(1)} P_{12}^{(1)} P_{21}^{(1)} P[X_0 = 2]$$

$$= \frac{3}{4}\frac{1}{4}\frac{3}{4}\frac{1}{3} = \frac{3}{64}$$

3.

From $P^2$, we get $P[X_2 = 1, X_0 = 0] = P_{01}^{(2)} = \frac{5}{16}$

$$P[X_2 = 1, X_0 = 0] = P[X_2 = 1/X_0 = 0]P[X_0 = 0]$$

$$= \frac{5}{16} \times \frac{1}{3} = \frac{5}{48}$$

**12.** A student's study habits are as follows. If he studies one night, he is 70% sure not to study next night. On the otherhand, the probability that he does not study two nights in succession is 0.6. In the long run, how often does he study?

**Solution**

The states of the system are S: Studying and T: Not Studying

The tpm of the Markov chain is

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

Let $\pi = (\pi_1 , \pi_2 )$ be the steady state distribution of the Markov chain. Then $\pi P = \pi.$

$$0.3\pi_1 + 0.34\pi_2 = \pi_1$$

$$0.7\pi_1 + 0.6\pi_2 = \pi_2$$

Also $\pi_1 + \pi_2 = 1$

Solving, we get,

$$\pi_1 = \frac{4}{11} , \quad \pi_2 = \frac{7}{11}$$

The steady state distribution of the given Markov chain is

$$\pi = \begin{bmatrix} \frac{4}{11} & \frac{7}{11} \end{bmatrix}$$

Thus is the long run, the student studies $\frac{4}{11}$ of the time.

**13.** A salesman territory consists of three cities A,B and C. He never sells in the same city on successive days. If he sells in city A, then the next day, he sells in city B. However, if he sells in either B or C, the next day he is twice as likely to sell in city A as in the other city. In the long run, how often does he sell in each of the cities?

**Solution**

The tpm of the given problem is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

The given problem describes a Markov chain with three states as three cities A,B,C.

We require the steady distribution of Markov chain for finding the probabilities in the long run.

Let the steady state probability distribution be $\pi = [\pi_0 \quad \pi_1 \quad \pi_2]$

Then $\pi P = \pi$.

$$[\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$$

$\frac{2}{3}(\pi_1 \quad + \pi_2) = \pi_1 \Rightarrow 3\pi_1 - 2\pi_2 - 2\pi_3 = 0$ (1)

$\pi_1 + \frac{\pi_3}{3} = \pi_2 \Rightarrow 3\pi_1 - 3\pi_2 + \pi_3 = 0$ (2)

$\frac{\pi_2}{3} = \pi_3 \Rightarrow \pi_2 = 3\pi_3$ (3)

Form (1) and (2), $3\pi_1 - 8\pi_3 = 0 \Rightarrow \pi_1 = \frac{8\pi_3}{3}$

$$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \frac{8\pi_3}{3} + 3\pi_3 + \pi_3 = 1$$

$$\frac{20\pi_3}{3} = 1 \Rightarrow \pi_3 = \frac{3}{20}$$

$$\pi_1 = \frac{8}{20}, \quad \pi_2 = \frac{9}{20}, \quad \pi_3 = \frac{3}{20}$$

The steady state probability distribution is

$$\pi = \left[\frac{8}{20}, \frac{9}{20}, \frac{3}{20}\right] = [0.40, 0.45, 0.15]$$

Thus in the long run, he sells 40% of the time in city A, 45% of the time in the city B and 15% of the time in city C.

**14.** A Psychologist makes the following assumptions concerning the behavior of mice subjected to a particular feeding schedule. For any particular trial, 80% of the mice that went right on the pervious experiment will go right on this trial and 60%of those mice that went left n the previous experiment will go right on this trial. If 50% went right on the first trial, what would he predict for (i) the second trial (ii) the third trial (iii) the thousandth trial?

**Solution**

The states of the system are R(right) and L(Lift). The transition matrix is

$$P = \begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix}$$

The probability distribution for the first trial is $P^{(1)} = (0.5 \;\; 0.5)$

To compute the probability distribution for the next step i.e. in the second trial, we compute $P^{(2)} = P^{(1)}P$

$$P^{(2)} = (0.5 \;\; 0.5)\begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} = (0.70 \;\; 0.3)$$

Thus on the second trial, he predicts that 70% of the mice will go right and 30% will go left. The probability distribution for the third trial is

$$P^{(3)} = P^{(2)}P = (0.70 \;\; 0.3)\begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} = (0.74 \;\; 0.26)$$

Thus on the third trial, he predicts that 74% of the mice will go right and 26% will go left.

We assume that the probability distribution of the Markov chain for the thousandth is essentially the steady state probability distribution.

Let $\pi = (\pi_1, \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1 \;\; \pi_2]\begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} = [\pi_1 \;\; \pi_2]$$

$$2\pi_1 - 6\pi_2 = 0$$

$$\pi_1 = 3\pi_2$$

Also $\pi_1 + \pi_2 = 1$

Solving, we get,

$$\pi_1 = \frac{3}{4}, \qquad \pi_2 = \frac{1}{4}$$

$$\pi = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

Thus he predicts that , on the thousandth trial, 75% of the mice will go to the right and 25% will go to the left.

**15.** Suppose that the probability of a dry day (state 0) following a rainy day(state 1) is $\frac{1}{3}$ and that the probability of a rainy day following a dry day is $\frac{1}{2}$. Given that May 1 is a dry day, find the probability that (i) May 3 is also a dry day (ii) May 5 is also a dry day.

**Solution**

This is a two state Markov chain with the probabilities $p_{10} = \frac{1}{3}$ and $p_{01} = \frac{1}{2}$.

State set ={0,1} where 0: Dry day and 1: Rainy day

The tpm of the Markov chain is $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

First let us compute $P^2$, $P^3$ and $P^4$

$$P^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{bmatrix} \quad P^3 = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \\ \frac{43}{108} & \frac{65}{108} \end{bmatrix} \quad P^4 = \begin{bmatrix} \frac{173}{432} & \frac{259}{432} \\ \frac{259}{648} & \frac{389}{648} \end{bmatrix}$$

Given that May 1 is a dry day

The probability that May 3 is also a dry day = $P_{00}^{(2)} = \frac{5}{12}$

The probability that May 5 is also a dry day = $P_{00}^{(4)} = \frac{173}{432}$

Reference

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