Poisson Process

If $X(t)$ represents the number of occurrences of a certain event in $(0,t)$ then the discrete random process $\{X(t)\}$ is called the Poisson process, provided the following postulates are satisfied.

1. $P\left(1 \text{ occurrence in } (t, t+\Delta t)\right) = \lambda \Delta t + O(\Delta t)$
2. $P\left(0 \text{ occurrence in } (t, t+\Delta t)\right) = 1 - \lambda \Delta t + O(\Delta t)$
3. $P\left(2 \text{ or more occurrence in } (t, t+\Delta t)\right) = O(\Delta t)$
4. $X(t)$ is independent of the number of occurrence of the event in any interval before and after interval $(0,t)$
5. The probability that any event occurs a specified number of times in $(t_0, t_0+\tau)$ depends only on τ and not on t_0.

Probability law for the Poisson process $\{X(t)\}$

$$P_n(t) = P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n = 0,1,2,3,\dots\dots$$

Proof:

Let λ be the number of occurrences of event in unit time.

Let $P_n(t) = P(X(t) = n) = \text{Probability of n occurrences in } (0,t)$

$$P_n(t+\Delta t) = P(X(t+\Delta t) = n)$$

$$= P(n-1 \text{ occurrences in } (0,t) \text{and } 1 \text{ occurrence in } (t, t+\Delta t)$$

$$+ P(n \text{ occurrences in } (0,t) \text{and } 0 \text{ occurrence in } (t, t+\Delta t))$$

$$= P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t), \text{ neglecting } O(\Delta t) \text{ terms}$$

$$= P_{n-1}(t)\lambda\Delta t + P_n(t) - P_n(t)\lambda\Delta t$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda(P_{n-1}(t)\Delta t - P_n(t))$$

Taking limit as $\Delta t \to 0$, we get

$$P_n'(t) = \frac{d}{dt}P_n(t) = \lambda(P_{n-1}(t)\Delta t - P_n(t)) \quad (1)$$

Let the solution of (1) be

$$P_n(t) = \frac{(\lambda t)^n}{n!}f(t) \quad (2)$$

Differentiate (2), with respect to *t*

$$P'_n(t) = \frac{(\lambda t)^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] \quad (3)$$

Using (2) and (3) in (1)

$$\frac{(\lambda t)^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] = \lambda\left[\frac{(\lambda t)^{n-1}}{(n-1)!}f(t) - \frac{(\lambda t)^n}{n!}f(t)\right]$$

$$\frac{\lambda^n t^{n-1}}{(n-1)!}f(t) + \frac{\lambda^n t^n}{n!}f(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}f(t) - \frac{\lambda(\lambda t)^n}{n!}f(t)$$

i.e.

$$\frac{\lambda^n t^n}{n!}f'(t) = -\frac{\lambda(\lambda t)^n}{n!}f(t)$$

$$f'(t) = -\lambda f(t)$$

$$\frac{f'(t)}{f(t)} = -\lambda$$

Integrating, we get $\log(f(t)) = -\lambda t + c$

$$f(t) = e^{-\lambda t + c} = ke^{-\lambda t} \quad (4)$$

Where $k = e^c$

Now

$$P_n(t) = \frac{(\lambda t)^n}{n!} f(t)$$

$$P_0(t) = f(t)$$

$$P_0(0) = f(0)$$

i.e. using (4), we have

$$P(X(0) = 0) = ke^{-0}$$

$$1 = k$$

(since P(no event occurs in (0,0)=1)

Substituting in (4) we get

$$f(t) = e^{-\lambda t}$$

By equation (2)

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n = 0,1,2,3, \dots \dots$$

Note

λ can be a constant or a function of *t*. In the above derivation, λ is assumed to be a constant. If λ is a constant, the process is called a homogeneous Poisson process.

Mean of the Poisson process

$$\text{Mean} = E\{X(t)\}$$

$$= \sum_{n=0}^{\infty} n P_n(t)$$

$$= \sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t} \left[\frac{\lambda t}{0!} + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots\right]$$

$$= e^{-\lambda t} \lambda t \left[1 + \frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t$$

Mean $= \lambda t$

Variance of Poisson process

$$E(X^2(t)) = \sum_{n=0}^{\infty} n^2 P_n(t)$$

$$= \sum_{n=0}^{\infty} [n(n-1) + n] \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{(\lambda t)^n e^{-\lambda t}}{n!} + \sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= e^{-\lambda t} \sum_{n=2}^{\infty} \frac{(\lambda t)^n}{(n-2)!} + \lambda t$$

$$= e^{\lambda t} \left[\frac{(\lambda t)^2}{0!} + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots\right] + \lambda t$$

$$= e^{\lambda t} (\lambda t)^2 \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right] + \lambda t$$

$$= e^{\lambda t} (\lambda t)^2 e^{-\lambda t} + \lambda t$$

$= \lambda t(\lambda t + 1)$

$$\text{Var}\{X(t)\} = E\{X^2(t)\} - (E\{X(t)\})^2$$

$$= \lambda t(\lambda t + 1) - (\lambda t)^2 = \lambda t$$

Note

Since mean $\{X(t)\}$ is not a constant, the Poisson process is not stationary.

Autocorrelation of the Poisson process

$$R_{XX}(t_1,t_2) = E(X(t_1)X(t_2))$$

$$= E[X(t_1)\big(X(t_2) - X(t_1)\big) + (X(t_1))]^2$$

$$= E[X(t_1)E\big(X(t_2) - X(t_1)\big) + E(X(t_1))]^2$$

(Since $\{X(t)\}$ is a process of independent increments)

$$= \lambda t_1(\lambda t_2 - \lambda t_1) + (\lambda t_1)^2 + \lambda t_1 \quad \text{if } t_2 > t_1$$

$$= \lambda^2 t_1 t_2 - \lambda^2 t_1{}^2 + \lambda^2 t_1{}^2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

$$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Auto covariance

Auto covariance = $C_{XX}(t_1, t_2)$

$$= R_{XX}(t_1,t_2) - E(X(t_1)X(t_2))$$

$$= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) - \lambda t_1 t_2$$

$$= \lambda \min(t_1, t_2)$$

Correlation coefficient

Correlation coefficient $= \frac{C_{XX}(t_1,t_2)}{\sqrt{Var(X(t_1))var(X(t_2))}}$

$$= \frac{\lambda \min(t_1,t_2)}{\sqrt{\lambda t_1 \lambda t_2}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda^2 t_1 \lambda t_2}}$$

$$= \sqrt{\frac{t_1}{t_2}}$$

Second order Poisson process

Second order probability function of a homogeneous Poisson process.

$= P(X(t_1) = n_1, X(t_2) = n_2)$ (assume $t_1 < t_2$)

$$= P(X(t_1) = n_1)P(X(t_2) = n_2/X(t_1) = n_1)$$

$$= P(X(t_1) = n_1)P(X(t_2 - t_1) = n_2 - n_1)$$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!} \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^{n_2-n_1}}{(n_2-n_1)!}$$

$= \frac{e^{-\frac{\lambda}{2}} \lambda^2 t_1^{n_1} (t_2-t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}$ if $n_2 \geq n_1$

0 otherwise

Similarly, the third order probability function is

$p(X(t_1) = n_1, p(X(t_2) = n_2, p(X(t_3) = n_3)$

$= e^{-\lambda t_2} \lambda^{n_3} e^{-\frac{\lambda}{2}} \lambda^2 t_1^{n_1} (t_2 - t_1)^{(n_2-n_1)} (t_3 - t_2)^{(n_3-n_2)}$ if $n_3 \geq n_2 \geq n_1$

0 otherwise

Properties of Poisson process

1 The Poisson process is a Markov process

Proof

$p(X(t_3) = n_3/ X(t_1) = n_1 \ \ and \ \ X(t_3) = n_3)$

$$= \frac{P(X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3)}{P(X(t_1)=n_1, X(t_2)=n_2)}$$

$$= \frac{e^{-\lambda t_2}\lambda^{n_3} e^{-\frac{\lambda}{2}} \lambda^2 t_1^{n_1} (t_2-t_1)^{(n_2-n_1)} (t_3-t_2)^{(n_3-n_2)}}{n_1!(n_2-n_1)!(n_3-n_2)!} \frac{n_1!(n_2-n_1)!}{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1} (t_2-t_1)^{(n_2-n_1)}}$$

(by second and third order probability functions)

$$= \frac{e^{-\lambda t_2}\lambda^{n_3} (t_3-t_2)^{(n_3-n_2)}}{e^{-\lambda t_2}\lambda^{n_2}(n_3-n_2)!}$$

$$= P(X(t_3) = n_3/X(t_2) = n_2)$$

(by comparison with the derivation of 2nd order probability function)

Thus, the conditional probability distribution of $X(t_3)$ given $X(t_1)$ and $X(t_2)$ depends only on the latest value $X(t_3) = n_2$. Hence the Poisson process is a Markov process.

2 The sum of two independent Poisson process is a Poisson process.

Proof.

Let $X(t) = X_1(t) + X_2(t)$

Where

$$P(X_1(t) = n) = \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!}, \qquad n = 0,1,2,3,\dots\dots$$

$$P(X_2(t) = n) = \frac{(\lambda_2 t)^n e^{-\lambda_2 t}}{n!}, \qquad n = 0,1,2,3,\dots\dots$$

$$P(X(t) = n) = \sum_{r=0}^{n} P(X_1(t) = r)P(X_1(t) = n - r)$$

By independence of $X(t_1)$ and $X(t_2)$

$$= \sum_{r=0}^{n} \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!} \frac{(\lambda_2 t)^{n-r} e^{-\lambda_2 t}}{(n-r)!}$$

$$= e^{-\lambda_1 t} e^{-\lambda_2 t} \sum_{r=0}^{n} \frac{(\lambda_1 t)^n}{n!} \frac{(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-t(\lambda_1+\lambda_2)} \sum_{r=0}^{n} n_{c_r} \frac{(\lambda_1 t)^n (\lambda_2 t)^{n-r}}{n!}$$

$$= \frac{e^{-t(\lambda_1+\lambda_2)} (\lambda_1 t + \lambda_2 t)^n}{n!}$$

$X_1(t) + X_2(t)$ is a Poisson process with parameter $(\lambda_1 + \lambda_2)t$

3 The difference of two independent Poisson processes is not a Poisson process.

Proof

Let $X(t) = X_1(t) + X_2(t)$

Then $E\{X(t)\} = E(X_1(t)) - E(X_2(t))$

$= \lambda_1 t - \lambda_2 t$

$= (\lambda_1 - \lambda_2)t$

$E\{X^2(t)\} = E(X^2{}_1(t)) - X^2{}_2(t) - 2X_1(t)X_2(t))$

$= E(X^2{}_1(t)) - E(X^2{}_2(t)) - 2E\big(X_1(t)\big)E(X_2(t))$

By independence of $X(t_1)$ and $X(t_2)$

$$= (\lambda_1 t)^2 + \lambda_1 t + (\lambda_2 t)^2 - \lambda_2 t - 2\lambda_1 t \lambda_2 t$$

$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

This is not equal to $(\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$

Note that if $X(t)$ is a Poisson process with parameter λt, then
$E\{X^2(t)\} = (\lambda t)^2 + \lambda t$

Hence $X_1(t) - X_2(t)$ is not a Poisson process.

References

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