POISSON PROCESS CONTINUED

We have been discussing the properties of the poisson process from the previous lecture and the fourth and last property will be discussed on this lecture;

Properties of Poisson Process (concluded)

4 The inter arrival time of a Poisson (i.e. the interval between two successive occurrences of a Poisson process) with parameter λ follows an exponential distribution with mean $\frac{1}{\lambda}$

Proof

Let the inter arrival time between two successive occurrences E_i and E_{i+1} of the Poisson process be T. If E_i takes place at time t_i, then E_{i+1} takes place at time $t_i + T$. T is a continuous random variable and $P(T > t) = P(E_{i+1}\text{ did not occur}$

in $(t_i, t_i + T))$

$$= P(\text{No event occurss in the interval of length t})$$

$$= P(X(t) = 0)$$

$$= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

Hence

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{d}{dt}F_T(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) = \lambda e^{-\lambda t} \quad t > 0$$

Which is the pdf of the exponential distribution

Mean $= E(t) = \int_0^\infty \lambda e^{-\lambda t} dt$

$$= \lambda \left[t\left(\frac{e^{-\lambda t}}{-\lambda}\right) - (1)\left(\frac{e^{-\lambda t}}{\lambda^2}\right)\right]$$

$$= \lambda \left(\frac{1}{\lambda^2}\right) = \frac{1}{\lambda}$$

If the number of occurrences of an event E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrence of E has a constant probability p of being recorded and the recordings are independent of each other, then the number $\mathrm{N(t)}$ of the recorded occurrences in t is also a Poisson process with parameter $\lambda\mathrm{p}$.

Proof

P(
$\mathrm{N(t) = n) = \sum_{n=0}^{\infty} P(E\ occurs\ (n + r)\ time\ in\ t\ and\ n\ of\ them\ are\ recorded)}$

$= \sum_{r=0}^{\infty} P(X(t) = n + r)(n + r)C_n\, p^n q^r$

$=\sum_{r=0}^{\infty} \frac{(\lambda t)^{n+r} e^{-\lambda t}}{(n+r)!}(n + r)\ C_n p^n q^r$

$= e^{-\lambda t}(\lambda t)^n p^n \sum_{r=0}^{\infty} \frac{(\lambda t)^r (n+r)!}{(n+r)!n!r!} q^r$

$=\frac{e^{-\lambda t}(\lambda p t)^n}{n!}\sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{r!}$

$=\frac{e^{-\lambda t}(\lambda p t)^n}{n!} e^{\lambda q t}$

$=\frac{e^{-\lambda t(1-q)}(\lambda p t)^n}{n!}, \qquad n = 0,1,2,\dots$

i.e. $\mathrm{N(t)}$ is a Poisson with parameter $\lambda\mathrm{p}$.

$X_1(t)$ and $X_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then

$P(X_1(t) = X_1(t) + X_2(t) = n) = nC_k\, p^k q^{n-k}$

Where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}, \qquad q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

i.e. the conditional distribution $X_1(t)$ given $(X_1(t) + X_2(t))$ is binomial.

Proof.

$$P(X_1(t) = k/X_1(t) + X_2(t) = n)$$

$$= \frac{P(N_1(t) = k, N_1(t) + N_2(t) = n)}{P(N_1(t) + N_2(t) = n)}$$

$$= \frac{P(N_1(t) = k, N_2(t) = n - k)}{P(N_1(t) + N_2(t) = n)}$$

$$= \frac{P(N_1(t) = k)P(N_2(t) = n - k)}{P(N_1(t) + N_2(t) = n)}$$

Since $N_1(t)$ and $N_2(t)$ are independent.

$$= \frac{\frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{k!}\frac{(\lambda_2 t)^n e^{-\lambda_2 t}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^n}{n!}}$$

$$= \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{k!}\frac{(\lambda_2 t)^n e^{-\lambda_2 t}}{(n-k)!}\frac{n!}{e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^n}$$

$$= nC_k \frac{(\lambda_1 t)^k(\lambda_2 t)^{n-k}}{((\lambda_1+\lambda_2)t)^n}$$

$$= nC_k \frac{{\lambda_1}^k{\lambda_2}^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$$= nC_k (\frac{\lambda_1}{\lambda_1+\lambda_2})^k(\frac{\lambda^2}{\lambda_1+\lambda_2})^{n-k}$$

$$= nC_k p^k q^{n-k}$$

Where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $\quad q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

Solved problem

1. Let $X(t)$ be a Poisson process with arrival rate λ. Find $E\{(X(t) - X(s))^2\}$ for *t*>*s*.

Solution

$E\left(X^2(t)\right) = (\lambda t)^2 + \lambda t$

$E\left(X(t)\right) = \lambda t,\ \ Var\left(X(t)\right) = \ \lambda t,$

Now

$E\{\left(X(t) - X(s)\right)^2\} = E(X^2(t) + X^2(s) - 2X(t)X(s))$

$= E(X^2(t)) + E(X^2(s)) - 2E(X(t)X(s))$

$= (\lambda t)^2 + \lambda t + (\lambda s)^2 + \lambda s - 2E[\left(X(t) - X(s)\right)X(s) + X^2(s)]$

$= (\lambda t)^2 + \lambda t + (\lambda s)^2 + \lambda s - 2(\lambda t - \lambda s)\lambda s - 2(\lambda^2 s^2 + \lambda s)$

Since $\{X(t)\}$ is a process of independent increments.

$= (\lambda t)^2 - (\lambda s)^2 + \lambda t - \lambda s - 2\lambda^2 ts + 2\lambda^2 s^2$

$= (\lambda t)^2 - (\lambda s)^2 - 2\lambda^2 ts + \lambda(t - s) = (t - s)^2\lambda^2 + \lambda(t - s)$

Note :

We may also use

$E\left(X(t)X(s)\right) = \text{auto correlation}$

$= \lambda^2 ts + \lambda\ (\text{t}, \text{s})$

$= \lambda^2 ts + \lambda \text{s}$

2.Messages arrive at a telegraph office according to Poisson process with $\lambda = 3/\text{hour}$. What is the probability that not messages arrives between 8 A.M and 12 noon?

Solution

Given

$\lambda = 3/\text{hour}$

$t = 4\text{hour}$

$n = \text{no of arrivals} = 0$

Now

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(X(4) = 0) = \frac{(12)^0 e^{-12}}{0!}$$

$$= e^{-12} = 6.144 \times 10^{-6}$$

3.Patients arrive randomly and independently at a doctor's consulting room from 8.00 A.M at an average rate of 1 every 5 minutes. The waiting room can hold 12 persons. What is probability that the room will be full when the doctor arrives at 9A.M?

Solution

Given

$\lambda = 12/\text{hour}$

$t = 1\text{hour}$

$n = \text{no of arrivals} = 12$

Now

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(X(1) = 12) = \frac{(12)^{12} e^{-12}}{12!} = 0.1144$$

4.An office receives 3 calls per minute on an average. What is the probability of receiving (a) no call in a one-minute interval (b) at the most 3 calls in a 5-minute interval?

Solution

(a) Given

$\lambda = 3/\text{hour}$

$t = 1\text{hour}$

$n = \text{no of arrivals} = 0$

Now

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(X(1) = 0) = \frac{(3)^0 e^{-3}}{0!} = 0.0498$$

$$\begin{aligned} P(X(5) \leq 3) \\ &= P(X(5) = 0) + P(X(5) = 1) + P(X(5) = 2) \\ &+ P(X(5) = 3) \end{aligned}$$

$$= \frac{(15)^0 e^{-15}}{0!} + \frac{(15)^1 e^{-15}}{1!} + \frac{(15)^2 e^{-15}}{2!} + \frac{(15)^3 e^{-15}}{3!}$$

$$= e^{-15}\left[1 + 15 + \frac{15^2}{2} + \frac{15^3}{6}\right]$$

$$= e^{-15}[1 + 15 + 112.5 + 562.5]$$

$$= 0.0002114$$

5.A hospital receives an average of 3 emergency calls in a 10-minute interval. What is the probability that there are at the most 3 emergency calls in a 10-minutes interval?

Solution

Given

$$\lambda = \frac{3}{10} = 0.3/\text{minute}$$

$t = 10\text{hour}$

$$P(X(10) \leq 3) = \frac{(3)^0 e^{-3}}{0!} + \frac{(3)^1 e^{-3}}{1!} + \frac{(3)^2 e^{-3}}{2!} + \frac{(3)^3 e^{-3}}{3!}$$

$$= e^{-3}\left[1 + 3 + \frac{9}{2} + \frac{27}{6}\right] = 0.647$$

6.Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 minutes.

1. Exactly 4 customers arrive
2. Less than 4 customers arrive
3. More than 4 customers arrive

Solution

Given

$\lambda = 3/\text{hour}$

$t = 2\text{hour}$

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

1. $P(X(2) = 4) = \frac{(6)^4 e^{-6}}{4!} = 0.1338$

2. $P(X(2) < 4) = \frac{(6)^0 e^{-6}}{0!} + \frac{(6)^1 e^{-6}}{1!} + \frac{(6)^2 e^{-6}}{2!} + \frac{(6)^3 e^{-6}}{3!}$

$$= e^{-6}[1 + 6 + 18 + 36] = 0.1412$$

3. $P(X(2) > 4) = 1 - (0.1338 + 0.1512) = 0.715$

7.A machine goes out of order whenever a component part fails. The failure of this part is in accordance with a Poisson process with a mean rate of 1 per week.

1. Find the probability that 2 weeks have lapsed since the last failture.

2. If there are 5 spare parts of this component in an inventory and the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks.

Solution

Given

$\lambda = 1\text{week}$

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

1. $t = 2, \quad n = 0$

   $$P(X(2) = 0) = \frac{(2)^0 e^{-2}}{0!} = 0.1353$$

2. $t = 10$

   $$P(X(10) \le 5) = e^{-10}\left[\frac{10^0}{0!} + \frac{10^1}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!} + \frac{10^5}{5!}\right]$$

   $= 0.067$

8.If $\{N(t)\}$ is a Poisson process and $s < t$, then

$$P(N(s) = k/N(t) = n) = nc_r \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

Solution

$$P(N(s) = k/N(t) = n) = \frac{P(N(s) = k \text{ and } N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(N(s) = k \text{ and } N(t-s) = n-k)}{P(N(t) = n)}$$

$$= \frac{P(N(s) = k)\mathrm{P}(N(t-s) = n-k)}{P(N(t) = n)}$$

$$= \frac{(\lambda s)^k e^{-\lambda s}}{k!} \frac{(\lambda(t-s))^{n-k} e^{-\lambda(t-s)}}{(n-k)!} \frac{n!}{(\lambda t)^n e^{-\lambda t}}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{s^k (t-s)^{n-k}}{t^n}$$

$$= nc_r \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k}$$

$$= nc_r \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$$

9.A radioactive source emits particles at a rate of 5 per minute according to a Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in a 4 minute period.

Solution

Given

$\lambda = 5/\text{min}$

Probability of recording $= p = 0.6$

$t = 4, n = 10$

Let $N(t)$ be the number of particles recorded in time interval t.

$$P(N(t) = n) = \frac{(\lambda pt)^n e^{-\lambda pt}}{n!}$$

Here $\lambda pt = 5 \times 0.6 \times 4 = 12$

$$P(N(4) = 10) = \frac{(12)^{10} e^{-12}}{10!} = 0.104$$

10. Suppose that customers arrive at a counter according to a Poisson process with mean rate of 2 per minute. Find the probability that the interval between two successive arrivals is

1. More then 1 minute
2. 4 minutes or less

3. Between 1 and 2 minutes

Solution

Let T be the random variable denoting inter arrival time. By property (3)

$f_T(t) = \lambda e^{-\lambda t}$ and $P(T > t) = e^{-\lambda t}$

Here $\lambda = 2/\text{minute}$ $\qquad f_T(t) = 2e^{-2t}$

1. $P(T > 1) = \int_1^{\infty} 2e^{-2t} dt = 0.1353$

2. $P(T \leq 4) = \int_0^{4} 2e^{-2t} dt = 0.99967$

3. $P(1 \leq T \leq 2) = \int_1^{2} 2e^{-2t} dt = 0.117$

11. Suppose that customers arrive at a counter from Town A at the rate of 1 per minute and from Town B at the rate of 2 per minute according to two independent Poisson processes. Find the probability that the interval between two successive arrivals is more than 1 minute.

Solution:

The total arrivals at the counter (from either source) constitute a Poisson process with mean 3 per minute.

Inter arrival time follows exponential distribution with

$$f_T(t) = 3e^{-3t}, \qquad t \geq 0$$

$$P(T > 1) = \int_1^{\infty} 3e^{-3t} dt = 0.0498$$

12. Customers arrive at the complaint department of a store at the rate of 5 per hour for male customers and 10 per hour for female customers. If arrivals in each case follow Poisson process, calculate the probabilities that (i) atmost 4 male customers (ii) atmost 4 female customers will arrive in a 30 minute period.

Solution

Let λ be the mean arrival rate.

By Poisson process, probability that n customers arrive in a time interval of t hours is given by

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0,1,2,...$$

1. The mean arrival rate for males is $\lambda = 5$

$$P(X(t) = n) = \frac{(5t)^n e^{-5t}}{n!}, \quad n = 0,1,2,...$$

$P[\text{Atmost 4 male customers arrive in a 30 minute period}]$

$= P\left[X\left(\frac{1}{2}\right) \leq 4\right]$

$=$

$P\left[X\left(\frac{1}{2}\right) = 0\right] + P\left[X\left(\frac{1}{2}\right) = 1\right] + P\left[X\left(\frac{1}{2}\right) = 2\right] + P\left[X\left(\frac{1}{2}\right) = 3\right] + P\left[X\left(\frac{1}{2}\right) = 4\right]$

$= e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$

$= e^{-2.5}[3.5 + 3.125 + 2.6042 + 1.6276] = 0.8912$

2. Mean arrival rate for female customers $\lambda = 10$

$P[\text{Atmost 4 female customers arrive in a 30 minute period}]$

$$= P\left[X\left(\frac{1}{2}\right) \leq 4\right]$$

Where

$$P(X(t) = n) = \frac{(10\mathrm{t})^n e^{-10\mathrm{t}}}{n!}, \quad n = 0,1,2,\dots$$

Required probability = $e^{-5}\left[1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right]$

= e^{-5}[1+5+12.5+20.8333+26.0417]

= 0.4405

13. Assume that the number of messages input to a communication channel in an interval of duration t seconds is a Poisson process with mean rate $\lambda = 0.3.$ Compute (i) The probability that exactly 3 messages will arrive during a ten-second interval (ii) The probability that the number of messages arriving in an interval of duration five seconds is between three and seven.

Solution

By Poisson process, probability of n messages arrive in t seconds

$$P(X(t) = n) = \frac{(\lambda\mathrm{t})^n e^{-\lambda\mathrm{t}}}{n!}, \quad n = 0,1,2,\dots$$

1. P[exactly 3 messages arrive during 10 second interval]

$$= P[X(10) = 3]$$

$$= \frac{(.3 \times 10)^3 e^{-.3\times 10}}{3!} = 0.2240$$

2. $P[3 \leq X(5) \leq 7] = \sum_{n=3}^{7} \frac{(1.5)^n e^{-1.5}}{n!}$

$=e^{-1.5}\left[\frac{(1.5)^3}{3!}+\frac{(1.5)^4}{4!}+\frac{(1.5)^5}{5!}+\frac{(1.5)^6}{6!}+\frac{(1.5)^7}{7!}\right]$

= .022313[0.5625+0.2109+0.0633+0.0158+0.0034]

= 0.1910

14. If particles are emitted from a radioactive source at the rate of 20 per hour. Find (i) The probability that exactly 5 particles are emitted during a 15 a minute period and (ii) The probability that fewer than 3 particles are emitted during a 12 minute period.

Solution

Let $X(t)$ denote the number of particles emitted during t minutes. By Poisson process

$$P(X(t)=n)=\frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n=0,1,2,...$$

Here $\lambda=\frac{20}{60}=\frac{1}{3}$

$$P(X(t)=n)=\frac{(\lambda\frac{t}{3})^n e^{-\lambda\frac{t}{3}}}{n!}, \quad n=0,1,2,...$$

P[exactly 5 particles are emitted during a 15 minute period]

$=P[X(15)=5]$

$$=\frac{(5)^5 e^{-5}}{5!}=0.1755$$

P[fewer than 3 particles are emitted during a 15 minute period]

$=P[X(12) < 3] = \sum_{n=0}^{2} \frac{(4)^n e^{-4}}{n!}$

$=e^{-4}[1 + 4 + 8] = 02.381$

References

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