FINAL EXAMINATION

ATTEMPT ANY FIVE QUESTIONS

Question 1.

A sample of 10 is drawn randomly from a certain population. The sum of the squared deviations from the mean of the given sample is 50. Test the hypothesis that the variance of the population is 5 at 5 per cent level of significance.

Solution

$$n = 10$$

$$\Sigma\left(X_i - \bar{X}\right)^2 = 50$$

$$\therefore \qquad \sigma_s^2 = \frac{\Sigma\left(X_i - \bar{X}\right)^2}{n-1} = \frac{50}{9}$$

Take the null hypothesis as $H_0: \sigma_p^2 = \sigma_s^2$. In order to test this hypothesis, we work out the χ^2 value as under:

$$\chi^2 = \frac{\sigma_s^2}{\sigma_p^2}(n-1) = \frac{\frac{50}{9}}{5}(10-1) = \frac{50}{9} \times \frac{1}{5} \times \frac{9}{1} = 10$$

Degrees of freedom = (10 – 1) = 9.

The table value of χ^2 at 5 per cent level for 9 d.f. is 16.92. The calculated value of χ^2 is less than this table value, so we accept the null hypothesis and conclude that the variance of the population is 5 as given in the question.

Question 2.

The following information is obtained concerning an investigation of 50 ordinary shops of small size:

	Shops		*Total*
	In towns	*In villages*	
Run by men	17	18	35
Run by women	3	12	15
Total	20	30	50

Can it be inferred that shops run by women are relatively more in villages than in towns? Use χ^2 test.

Solution

Take the hypothesis that there is no difference so far as shops run by men and women in towns and villages. With this hypothesis the expectation of shops run by men in towns would be:

$$\text{Expectation of } (AB) = \frac{(A) \times (B)}{N}$$

where A = shops run by men
B = shops in towns
$(A) = 35$; $(B) = 20$ and $N = 50$

$$\text{Thus, expectation of } (AB) = \frac{35 \times 20}{50} = 14$$

Hence, table of expected frequencies would be

	Shops in towns	*Shops in villages*	*Total*
Run by men	14 (AB)	21 (Ab)	35
Run by women	6 (aB)	9 (ab)	15
Total	20	30	50

Calculation of χ^2 value:

Groups	*Observed frequency* O_{ij}	*Expected frequency* E_{ij}	$(O_{ij} - E_{ij})$	$(O_{ij} - E_{ij})^2/E_{ij}$
(AB)	17	14	3	9/14 = 0.64
(Ab)	18	21	–3	9/21 = 0.43
(aB)	3	6	–3	9/6 = 1.50
(ab)	12	9	3	9/9 = 1.00

$$\therefore \quad \chi^2 = \Sigma \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 3.57$$

As one cell frequency is only 3 in the given 2 × 2 table, we also work out χ^2 value applying Yates' correction and this is as under:

$$\chi^2(\text{corrected}) = \frac{[|17-14|-0.5]^2}{14} + \frac{[|18-21|-0.5]^2}{21} + \frac{[|3-6|-0.5]^2}{6} + \frac{[|12-9|-0.5]^2}{9}$$

$$= \frac{(2.5)^2}{14} + \frac{(2.5)^2}{21} + \frac{(2.5)^2}{6} + \frac{(2.5)^2}{9}$$

$$= 0.446 + 0.298 + 1.040 + 0.694$$

$$= 2.478$$

∵ Degrees of freedom = $(c-1)(r-1) = (2-1)(2-1) = 1$

Table value of χ^2 for one degree of freedom at 5 per cent level of significance is 3.841. The calculated value of χ^2 by both methods (i.e., before correction and after Yates' correction) is less than its table value. Hence the hypothesis stands. We can conclude that there is no difference between shops run by men and women in villages and towns.

Question 3.

A housewife buys 3 kinds of cereals, A, B and C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys B. However, if she buys B or C, the next time she is 3 times as likely to buy A as the other cereal. In the long ron, how often does she buy each of the three cereals?

Solution

The transition probability matrix of the process is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Let $\pi = (\pi_1, \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1, \pi_2, \pi_3]\begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\frac{3\pi_2}{4} + \frac{3\pi_3}{4} = \pi_1$$

$$\pi_1 + \frac{\pi_3}{4} = \pi_2$$

$$\frac{\pi_2}{4} = \pi_3$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$

Solving, we get,

$$\pi_1 = \frac{15}{35}, \quad \pi_2 = \frac{16}{35}, \quad \pi_3 = \frac{4}{35}$$

In the long run, probability of buying $A = \frac{15}{35}$

In the long run, probability of buying $B = \frac{16}{35}$

In the long run, probability of buying $C = \frac{4}{35}$

Question 4.

A man either drives a car or catches a train to go to the office each day. He never goes 2 days on a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a six appeared. Find:

1. The probability that he takes a train on the third day and,
2. The probability that he drives to work in the long run.

Solution

The state space of the Markov chain is { Train, Car}. The transition probability matrix is $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Now , on the first day,

$P(\text{driving to work}) = P(\text{getting 6}) = \frac{1}{6}$

$P(\text{taking a train}) = \frac{5}{6}$

The initial probability distribution is

$$P^{(1)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Now

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

The probability distribution for the third day

i.e.

$$P^{(3)} = P^{(1)} P^2$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{24} & \frac{13}{24} \end{bmatrix}$$

Probability that he takes a train on the third day $= \frac{11}{24}$

Let $\pi = (\pi_1 , \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$[\pi_1 \quad \pi_2] \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\frac{\pi_2}{2} = \pi_1$$

$$\pi_1 + \frac{\pi_2}{2} = \pi_2$$

Also $\qquad \pi_1 + \pi_2 = 1$

Solving, we get,

$$\pi_1 = \frac{1}{3} , \qquad \pi_2 = \frac{2}{3}$$

$$\pi = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The probability that the man drives to work in the long run $= \frac{2}{3}$

Question 5.

A student's study habits are as follows. If he studies one night, he is 70% sure not to study next night. On the otherhand, the probability that he does not study two nights in succession is 0.6. In the long run, how often does he study?

Solution

The states of the system are S: Studying and T: Not Studying

The tpm of the Markov chain is

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

Let $\pi = (\pi_1, \pi_2)$ be the steady state distribution of the Markov chain. Then $\pi P = \pi$.

$$0.3\pi_1 + 0.34\pi_2 = \pi_1$$

$$0.7\pi_1 + 0.6\pi_2 = \pi_2$$

Also $\pi_1 + \pi_2 = 1$

Solving, we get,

$$\pi_1 = \frac{4}{11}, \quad \pi_2 = \frac{7}{11}$$

The steady state distribution of the given Markov chain is

$$\pi = \begin{bmatrix} \frac{4}{11} & \frac{7}{11} \end{bmatrix}$$

Thus is the long run, the student studies $\frac{4}{11}$ of the time.

Question 6.

Patients arrive randomly and independently at a doctor's consulting room from 8.00 A.M at an average rate of 1 every 5 minutes. The waiting room can hold 12 persons. What is probability that the room will be full when the doctor arrives at 9A.M?

Solution

Given

$\lambda = 12/\text{hour}$

$t = 1\text{hour}$

$n = \text{no of arrivals} = 12$

Now

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$P(X(1) = 12) = \frac{(12)^{12} e^{-12}}{12!} = 0.1144$$

Question 7.

Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute. Find the probability that during a time interval of 2 minutes.

1. Exactly 4 customers arrive
2. Less than 4 customers arrive
3. More than 4 customers arrive

Solution

Given

$\lambda = 3/\text{hour}$

$t = 2\text{hour}$

$$P(X(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

1. $P(X(2) = 4) = \frac{(6)^4 e^{-6}}{4!} = 0.1338$

2. $P(X(2) < 4) = \frac{(6)^0 e^{-6}}{0!} + \frac{(6)^1 e^{-6}}{1!} + \frac{(6)^2 e^{-6}}{2!} + \frac{(6)^3 e^{-6}}{3!}$

$= e^{-6}[1 + 6 + 18 + 36] = 0.1412$

3. $P(X(2) > 4) = 1 - (0.1338 + 0.1512) = 0.715$