Fundamentals of Vector Spaces Continued

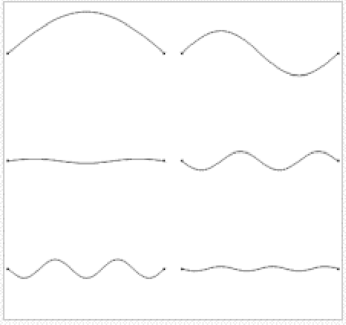
Inner Product Spaces and Hilbert Spaces

In mathematics, an **inner product space** or a **Hausdorff pre-Hilbert space** is a vector space with a binary operation called an **inner product.** This operation associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors, often denoted using angle brackets (as in $\langle a, b \rangle$). Inner products allow the rigorous introduction of intuitive geometrical notions, such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension, and are studied in functional analysis. Inner product spaces over the field of complex numbers are sometimes referred to as **unitary spaces**. The first usage of the concept of a vector space with an inner product is due to Giuseppe Peano, in 1898.

An inner product naturally induces an associated norm, (|*x*| and |*y*| are the norms of x and y, in the picture), which canonically makes every inner product space into a normed vector space. If this normed space is also a Banach space then the inner product space is called a **Hilbert space**.[1] If an inner product space $(H, \langle \cdot, \cdot \rangle)$ is not a Hilbert space then it can be "extended" to a Hilbert space $(\overline{H}, \langle \cdot, \cdot \rangle_{\overline{H}})$, called a completion. Explicitly, this means that H is linearly and isometrically embedded onto a dense vector subspace of $\overline{H}$ and that the inner product $\langle \cdot, \cdot \rangle_{\overline{H}}$ on $\overline{H}$ is the unique continuous extension of the original inner product

Hilbert Space

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space to spaces with any finite or infinite number of dimensions. A Hilbert space is a vector space equipped with an inner product, an operation that allows lengths and angles to be defined. Furthermore, Hilbert spaces are complete, which means that there are enough limits in the space to allow the techniques of calculus to be used.



The state of a vibrating string can be modeled as a point in a Hilbert space. The decomposition of a vibrating string into its vibrations in distinct overtones is given by the projection of the point onto the coordinate axes in the space.

Hilbert spaces arise naturally and frequently in mathematics and physics, typically as infinite-dimensional function spaces. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert, Erhard Schmidt, and Frigyes Riesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer), and ergodic theory (which forms the mathematical underpinning of thermodynamics). John von Neumann coined the term *Hilbert space* for the abstract concept that underlies many of these diverse applications. The success of Hilbert space methods ushered in a very fruitful era for functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analogs of the Pythagorean theorem and parallelogram law hold in a Hilbert space. At a deeper level, perpendicular projection onto a subspace (the analog of "dropping the altitude" of a triangle) plays a significant role in optimization problems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis), in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, the Hilbert space can also be usefully thought of in terms of the space of infinite sequences that are square-summable. The latter space is often in the older literature referred to as *the* Hilbert space. Linear operators on a Hilbert space are likewise fairly concrete objects: in good cases, they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectrum.

Inner Product Spaces and Hilbert Spaces

Similar to magnitude / length of a vector, another important concept in three dimensional space that needs to be generalized is angle between any two vectors. Given any two unit vectors in R^3, say $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$, the angle between these two vectors is defined using inner (or dot) product of two vectors as

$$\cos(\theta) = (\widehat{\mathbf{x}})^T \widehat{\mathbf{y}} = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)^T \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \qquad \text{-----------} \ (27)$$

$$= \widehat{x}_1 \widehat{y}_1 + \widehat{x}_2 \widehat{y}_2 + \widehat{x}_3 \widehat{y}_3 \qquad \text{-----------} \ (28)$$

The fact that cosine of angle between any two unit vectors is always less than one can be stated as

$$|\cos(\theta)| = |\langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle| \leq 1 \qquad \text{-----------} \ (29)$$

Moreover, vectors $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal if $(\mathbf{x})^T \mathbf{y} = 0$. Orthogonality is probably the most useful concept while working in three dimensional Euclidean space. Inner product spaces and Hilbert spaces generalize these simple geometrical concepts in three dimensional Euclidean space to higher or infinite dimensional vector spaces.

Definition 38 (Inner Product Space): An inner product space is a linear vector space X together with an inner product defined on $X \times X$. Corresponding to each pair of vectors $\mathbf{x}, \mathbf{y} \in X$ the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ of $\mathbf{x}$ and $\mathbf{y}$ is a scalar. The inner product satisfies following axioms.

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (complex conjugate)
2. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
3. $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \overline{\lambda} \langle \mathbf{x}, \mathbf{y} \rangle$

   $\langle \mathbf{x}, \lambda \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \overline{0}$.

Definition 39 (Hilbert Space): A complete inner product space is called as an Hilbert space.

Here are some examples of commonly used inner product and Hilbert spaces.

Example 40 Inner Product Spaces

1. $X = R^n$ with inner product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i \qquad \text{-----------} \ (30)$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} (x_i)^2 = \|\mathbf{x}\|_2^2 \qquad \text{-----------} \ (31)$$

is a Hilbert space.

2. $X = R^n$ with inner product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^T W \mathbf{y} \qquad \text{----------- (32)}$$

where W is a positive definite matrix is a Hilbert space. The corresponding 2-norm is defined as $\|\mathbf{x}\|_{W,2} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_W} = \sqrt{\mathbf{x}^T W \mathbf{x}}$

3. $X = C^n$ with inner product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \overline{x}_i y_i \qquad \text{----------- (33)}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} \overline{x}_i x_i = \sum_{i=1}^{n} |x_i|^2 = \|\mathbf{x}\|_2^2 \qquad \text{----------- (34)}$$

is a Hilbert space.

4. The set of real valued square integrable functions on interval $[a, b]$ with inner product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}(t)\mathbf{y}(t)dt \qquad \text{----------- (35)}$$

is an Hilbert space and denoted as $L_2[a, b]$. Well known examples of spaces of this type are the set of continuous functions on $L_2[-\pi, \pi]$ or $L_2[0, 2\pi]$, which are considered while developing Fourier series expansions of continuous functions on $[-\pi, \pi]$ or $[0, 2\pi]$ using $sin(n\pi)$ and $cos(n\pi)$ as basis functions.

5. Space of polynomial functions on $[a, b]$ with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{x}(t)\mathbf{y}(t)dt \qquad \text{----------- (36)}$$

is a inner product space. This is a subspace of $L_2[a, b]$.

6. Space of complex valued square integrable functions on $[a, b]$ with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \overline{\mathbf{x}}(t)\mathbf{y}(t)dt \qquad \text{----------- (37)}$$

is an inner product space.

Axioms 2 and 3 imply that the inner product is linear in the first entry. The quantity $\langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ is a candidate function for defining norm on the inner product space. Axioms 1 and 3 imply that $\|\alpha \mathbf{x}\| = |\alpha| \; \|\mathbf{x}\|$ and axiom 4 implies that $\|\mathbf{x}\| > 0$ for $\mathbf{x} \neq \overline{0}$. If we show that $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ satisfies triangle inequality, then $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ defines a norm on space X . We first prove Cauchy-Schwarz inequality, which is generalization of equation (cos), and proceed to show that $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ defines the well known 2-norm on X, i.e. $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Lemma 41 (**Cauchey- Schwarz Inequality):**

Let X denote an inner product space. For all $\mathbf{x},\mathbf{y} \in X$,the following inequality holds

$$|\langle \mathbf{x},\mathbf{y}\rangle| \leq [\langle \mathbf{x},\mathbf{x}\rangle]^{1/2}[\langle \mathbf{y},\mathbf{y}\rangle]^{1/2} \qquad \text{----------- (38)}$$

The equality holds if and only if $\mathbf{x} = \lambda\mathbf{y}$ or $\mathbf{y} = \overline{0}$

Proof: If $\mathbf{y} = \overline{0}$, the equality holds trivially so we assume $\mathbf{y} \neq \overline{0}$. Then, for all scalars λ, we have

$$0 \leq \langle \mathbf{x}-\lambda\mathbf{y},\mathbf{x}-\lambda\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle - \lambda\langle \mathbf{x},\mathbf{y}\rangle - \overline{\lambda}\langle \mathbf{y},\mathbf{x}\rangle + |\lambda|^2\langle \mathbf{y},\mathbf{y}\rangle \qquad \text{----------- (39)}$$

In particular, if we choose $\lambda = \dfrac{\langle \mathbf{y},\mathbf{x}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle}$, then, using axiom 1 in the definition of inner product, we have

$$\overline{\lambda} = \frac{\overline{\langle \mathbf{y},\mathbf{x}\rangle}}{\langle \mathbf{y},\mathbf{y}\rangle} = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle} \qquad \text{----------- (40)}$$

$$\begin{aligned} &\Rightarrow -\lambda\langle \mathbf{x},\mathbf{y}\rangle - \overline{\lambda}\langle \mathbf{y},\mathbf{x}\rangle = -\frac{2\langle \mathbf{x},\mathbf{y}\rangle\,\langle \mathbf{y},\mathbf{x}\rangle}{\langle \mathbf{y},\mathbf{y}\rangle} \\ &= -\frac{2\langle \mathbf{x},\mathbf{y}\rangle\overline{\langle \mathbf{x},\mathbf{y}\rangle}}{\langle \mathbf{y},\mathbf{y}\rangle} = -\frac{2|\langle \mathbf{x},\mathbf{y}\rangle|^2}{\langle \mathbf{y},\mathbf{y}\rangle} \end{aligned} \qquad \text{----------- 41)}$$

$$\Rightarrow 0 \leq \langle \mathbf{x},\mathbf{x}\rangle - \frac{|\langle \mathbf{x},\mathbf{y}\rangle|^2}{\langle \mathbf{y},\mathbf{y}\rangle} \qquad \text{----------- (42)}$$

$$\text{or } |\langle \mathbf{x},\mathbf{y}\rangle| \leq \sqrt{\langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle} \qquad \text{----------- (43)}$$

The triangle inequality can be can be established easily using the Cauchy-Schwarz inequality as follows

$$\text{----------- (44)}$$

$$\begin{aligned} \langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle &= \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle. && \text{----------- (45)} \\ &\leq \langle \mathbf{x},\mathbf{x}\rangle + 2|\langle \mathbf{x},\mathbf{y}\rangle| + \langle \mathbf{y},\mathbf{y}\rangle \\ &\leq \langle \mathbf{x},\mathbf{x}\rangle + 2\sqrt{\langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle} + \langle \mathbf{y},\mathbf{y}\rangle && \text{----------- (46)} \end{aligned}$$

$$\sqrt{\langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle} \leq \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} + \sqrt{\langle \mathbf{y},\mathbf{y}\rangle} \qquad \text{----------- (47)}$$

Thus, the candidate function $\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$ satisfies all the properties necessary to define a norm, i.e.

$$\begin{aligned} & && \text{----------- (48)} \\ &\sqrt{\langle \mathbf{x},\mathbf{x}\rangle} \geq 0 \ \forall\, \mathbf{x} \in X \text{ and } \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} = 0 \ \mathit{iff}\ \mathbf{x} = \overline{0} \\ & && \text{----------- (49)} \\ &\sqrt{\langle \alpha\mathbf{x},\alpha\mathbf{x}\rangle} = |\alpha|\sqrt{\langle \mathbf{x},\mathbf{x}\rangle} \\ &\sqrt{\langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle} \leq \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} + \sqrt{\langle \mathbf{y},\mathbf{y}\rangle} \qquad \text{(Triangle inequality)} && \text{----------- (50)} \end{aligned}$$

Thus, the function $\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$ indeed defines a norm on the inner product space X. In fact the inner product defines the well known 2-norm on X, i.e.

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} \quad \text{-----------} \ (51)$$

and the triangle inequality can be stated as

$$\|\mathbf{x}+\mathbf{y}\|_2^2 \le \|\mathbf{x}\|_2^2 + 2\|\mathbf{x}\|_2 . \|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2 . = [\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2]^2 \quad \text{-----------} \ (52)$$

$$\text{or } \|\mathbf{x}+\mathbf{y}\|_2 \le \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 \quad \text{-----------} \ (53)$$

Definition 42 (**Angle**) The angle θ between any two vectors in an inner product space is defined by

$$\theta = \cos^{-1}\left[\frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}\right] \quad \text{-----------} \ (54)$$

Definition 432 (Orthogonal Vectors): In a inner product space X two vector $\mathbf{x},\mathbf{y} \in X$ are said to be orthogonal if $\langle \mathbf{x},\mathbf{y}\rangle = 0$. We symbolize this by $\mathbf{x}\perp\mathbf{y}$. A vector $\mathbf{x}$ is said to be orthogonal to a set S (written as $\mathbf{x}\perp S$) if $\mathbf{x}\perp\mathbf{z}$ for each $\mathbf{z} \in S$.

Just as orthogonality has many consequences in three dimensional geometry, it has many implications in any inner-product / Hilbert space Luen. The Pythagoras theorem, which is probably the most important result the plane geometry, is true in any inner product space.

Lemma 44 If $\mathbf{x}\perp\mathbf{y}$ in an inner product space then $\|\mathbf{x}+\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2$.

Proof: $\|\mathbf{x}+\mathbf{y}\|_2^2 = \langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2 + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle$.

Definition 45 (Orthogonal Set): A set of vectors S in an inner product space X is said to be an orthogonal set if $\mathbf{x}\perp\mathbf{y}$ for each $\mathbf{x},\mathbf{y} \in S$ and $\mathbf{x} \neq \mathbf{y}$. The set is said to be orthonormal if, in addition each vector in the set has norm equal to unity.

Note that an orthogonal set of nonzero vectors is linearly independent set. We often prefer to work with an orthonormal basis as any vector can be uniquely represented in terms of components along the orthonormal directions. Common examples of such orthonormal basis are (a) unit vectors along coordinate directions in R^n (b) function $\{sin(nt) : n = 1,2,...\}$ and $\{cos(nt) : n = 1,2,...\}$ in $L_2[0,2\pi]$.

Example 46 Show that function $\langle \mathbf{x},\mathbf{y}\rangle_W : R^n \times R^n \rightarrow R$ defined as

$$\langle \mathbf{x},\mathbf{y}\rangle_W = \mathbf{x}^T W\mathbf{y}$$

defines an inner product on when W is a symmetric positive definite matrix.

Solution: For $\langle \mathbf{x},\mathbf{y}\rangle_W = x^T Wy$ to qualify as inner product, it must satisfy the following all four axioms in the definition of the inner product. We have,

$$\langle \mathbf{x},\mathbf{y}\rangle_W = \mathbf{x}^T W\mathbf{y} \ \text{and} \ \langle \mathbf{y},\mathbf{x}\rangle_W = \mathbf{y}^T W\mathbf{x}$$

Since W is symmetric, i.e.

$$W^T = W, \ \left[\mathbf{x}^T W\mathbf{y}\right]^T = \mathbf{y}^T W^T\mathbf{x} = \mathbf{y}^T W\mathbf{x}$$

Thus, axiom A1 holds for any $x,y \in R^n$.

$$\langle \mathbf{x}+\mathbf{y},\mathbf{z}\rangle_W = (\mathbf{x}+\mathbf{y})^T W\mathbf{z} = \mathbf{x}^T W\mathbf{z} + \mathbf{x}^T W\mathbf{z} = \langle \mathbf{x},\mathbf{z}\rangle_W + \langle \mathbf{y},\mathbf{z}\rangle_W$$

Thus, axiom A2 holds for any $x, y, z \in R^n$.

$$\langle \lambda\mathbf{x},\mathbf{y}\rangle = (\lambda\mathbf{x})^T W\mathbf{y} = \lambda(\mathbf{x}^T W\mathbf{y}) = \lambda\langle \mathbf{x},\mathbf{y}\rangle$$
$$\langle \mathbf{x},\lambda\mathbf{y}\rangle = \mathbf{x}^T W(\lambda\mathbf{y}) = \lambda(\mathbf{x}^T W\mathbf{y}) = \lambda\langle \mathbf{x},\mathbf{y}\rangle$$

Thus, axiom A3 holds for any $\mathbf{x},\mathbf{y} \in R^n$. Since W is positive definite, it follows that $\langle \mathbf{x},\mathbf{x}\rangle_W = \mathbf{x}^T W\mathbf{x} > 0$ if x $\neq \overline{\mathbf{0}}$ and $\langle \mathbf{x},\mathbf{x}\rangle_W = \mathbf{x}^T W\mathbf{x} = 0$ if x $= \overline{\mathbf{0}}$. Thus, axiom A4 holds for any $\mathbf{x} \in R^n$. Since all four axioms are satisfied, $\langle \mathbf{y},\mathbf{x}\rangle_W = \mathbf{y}^T W\mathbf{x}$ is a valid definition of an inner product.

Example 47 The triangle inequality asserts that, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ belonging to an inner product space

$$\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{y}\|_2 + \|\mathbf{x}\|_2$$

Does the Cauchy-Schwartz inequality follow from the triangle inequality? Under what condition Schwartz inequality becomes an equality?

Solution: Squaring both the sides, we have

$$\|\mathbf{x}+\mathbf{y}\|_2^2 = \langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle \leq [\|\mathbf{y}\|_2 + \|\mathbf{x}\|_2]^2$$

$$\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle + 2\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2\|\mathbf{x}\|_2$$

$$\|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 + 2\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2\|\mathbf{x}\|_2$$

Since, $\|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 \geq 0$ for any $\mathbf{x},\mathbf{y} \in X$, the above inequality reduces to

$$\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2\|\mathbf{x}\|_2 \quad \text{-----------} \ (55)$$

The triangle inequality also implies that

$$\|\mathbf{x}-\mathbf{y}\|_2^2 = \langle \mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle \leq [\|\mathbf{y}\|_2 + \|\mathbf{x}\|_2]^2$$

$$\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle - 2\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2\|\mathbf{x}\|_2$$

$$\|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 - 2\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 + 2\|\mathbf{y}\|_2\|\mathbf{x}\|_2$$

Since, $\|\mathbf{y}\|_2^2 + \|\mathbf{x}\|_2^2 \geq 0$ for any $\mathbf{x},\mathbf{y} \in X$, the above inequality reduces to

$$-\langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2\|\mathbf{x}\|_2$$

i.e.

$$-\|\mathbf{y}\|_2\|\mathbf{x}\|_2 \leq \langle \mathbf{x},\mathbf{y}\rangle \quad \text{-----------} \ (56)$$

Combining inequalities (R1) and (R2), we arrive at the Cauchy-Schwartz inequality

$$-\|\mathbf{y}\|_2\|\mathbf{x}\|_2 \leq \langle \mathbf{x},\mathbf{y}\rangle \leq \|\mathbf{y}\|_2\|\mathbf{x}\|_2 \quad \text{-----------} \ (57)$$

i.e.

$$|\langle \mathbf{x},\mathbf{y}\rangle| \leq \|\mathbf{y}\|_2\|\mathbf{x}\|_2 \quad \text{-----------} \ (58)$$

The Cauchy-Schwartz inequality reduces to equality when $\mathbf{y} = \alpha\mathbf{x}$.

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