Problem Discretization using Approximation Theory

Unified Problem Representation

In the previous, we have listed and categorized different types of equations that arise in variety of engineering problems. The fundamentals of vector spaces were introduced in the subsequent module. With this background, we are ready to start our journey in the numerical analysis. We first show that the concept of vector space allows us to develop a unified representation of seemingly different problems, which were initially categorized as algebraic equations, ODE-IVPs, ODE-BVP, PDEs etc., as a transformation of a vector from one vector space to another. When the transformations involved in a problem at hand are non-linear, it is often not possible to solve the problem analytically. In all such cases, the problem is approximated and transformed to a computationally tractable form, i.e.,

$$\begin{bmatrix} \text{Original} \\ \text{Problem} \end{bmatrix} \xrightarrow{\text{Approximation}} \begin{bmatrix} \text{Computationally Tractable} \\ \text{Approximation} \end{bmatrix}$$

and we compute an approximate solution using the computable version. Figure SolnScheme presents a schematic representation of how a numerical solution scheme is formulated for a problem at hand. It may be noted that the problem is transformed to one of the standard computable forms and then one or more stadard tools are used to construct approximate solution of the original problem. In some way, a numerical solution scheme can be considered analogous to a measuring instrument, which generates a *reasonable approximation* of a measured physical variable in a transformed domain. The measurements are acceptable as long as the errors in approximation are *small*. In this module, we explain the process of problem approximation using various approaches available in the literature. In the end, we distill out generic equation forms that frequently arise in the process of the problem approximation.

1. Unified Problem Representation

Using the generalized concepts of vectors and vector spaces discussed in the previous module, we can look at mathematical models in engineering as transformations, which map a subset of vectors from one vector space to a subset in another space.

Definition 1 (**Transformation):** Let X and Y be linear spaces and let M be subset of X. A rule which associates with every element $\mathbf{x} \in M$ to an element $\mathbf{y} \in Y$ is said to be transformation from X to Y with domain M. If $\mathbf{y}$ corresponds to $\mathbf{x}$ under the transformation we write $\mathbf{y} = \mathcal{T}(\mathbf{x})$ where $\mathcal{T}(.)$ is called an operator.

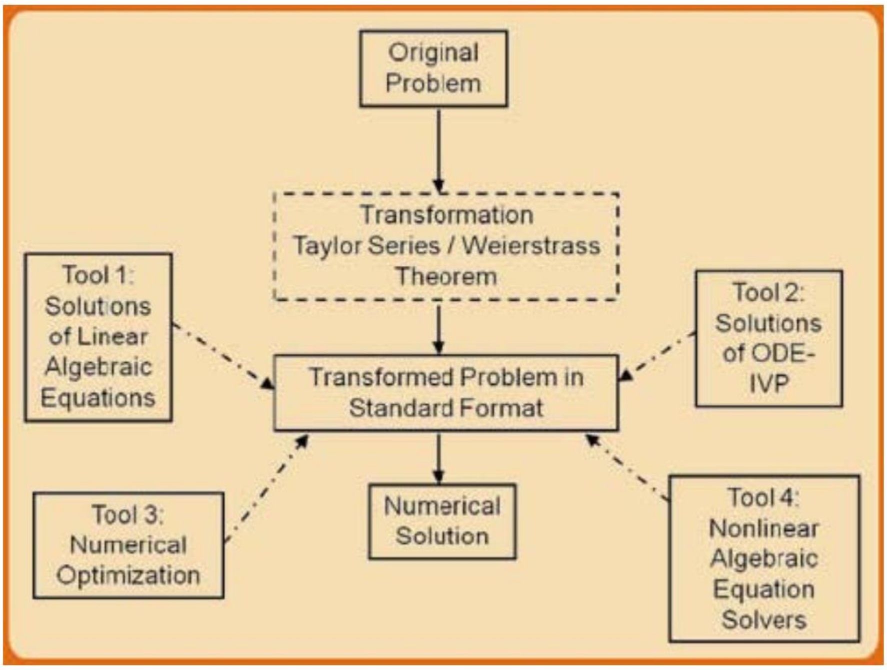

Figure 1: Formulation of Numerical Solution Scheme

The set of all elements for which an operator $\mathcal{T}$ is defined is called as domain of $\mathcal{T}$ and the set of all elements generated by transforming elements in the domain by $\mathcal{T}$ are called as range of $\mathcal{T}$. If for every $\mathbf{y} \in Y$, there is utmost one $\mathbf{x} \in M$ for which $\mathcal{T}(\mathbf{x}) = \mathbf{y}$, then $\mathcal{T}(.)$ is said to be *one to one*. If for every $\mathbf{y} \in Y$ there is at least one $\mathbf{x} \in M$, then $\mathcal{T}$ is said to map M *onto* Y. A transformation is said to be invertible if it is *one to one* and *onto*.

Definition 2 (Linear Transformations): A transformation $\mathcal{T}$ mapping a vector space X into a vector space Y is said to be linear if **for every** $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in X$ and all scalars α, β we have

$$\mathcal{T}(\alpha x^{(1)} + \beta x^{(2)}) = \alpha \mathcal{T}(\mathbf{x}^{(1)}) + \beta \mathcal{T}(\mathbf{x}^{(2)}). \qquad \text{-------- (1)}$$

Note that any transformation that does not satisfy the above definition is not a linear transformation.

Definition 3 (Continuous Transformation): A transformation $\mathcal{T}: M \to Y$ is continuous at point $\mathbf{x}^* \in M$ if and only if $\{\mathbf{x}^{(n)}\} \to \mathbf{x}^*$ implies $\mathcal{T}(\mathbf{x}^{(n)}) \to \mathcal{T}(\mathbf{x}^*)$. If $\mathcal{T}(.)$ is continuous at **each** $\mathbf{x}^* \in M$, then we say that the function is a continuous function on M.

Example 4 Operators

1. Consider transformation

$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \text{-------- (2)}$$

where $\mathbf{y} \in \mathbf{R}^m, \mathbf{x} \in \mathbf{R}^n$, $A \in \mathbf{R}^m \times \mathbf{R}^n$ and $\mathcal{T}(\mathbf{x}) = A\mathbf{x}$. Whether this mapping is onto R^m depends on the rank of the matrix. It is easy to check that A is a linear operator.

2. Consider transformation

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad \text{-------- (3)}$$

where $\mathbf{y}, \mathbf{b} \in \mathbf{R}^m, \mathbf{x} \in \mathbf{R}^n$, $A \in \mathbf{R}^m \times \mathbf{R}^n$ and $\mathcal{T}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. Here, $\mathbf{b}$ is a fixed non-zero vector. Note that this transformation does not satisfy equation (1) and does not qualify as a linear transformation.

3. Consider transformation involving differentiation, i.e.

$$y(t) = \frac{dx(t)}{dt}$$

where $t \in [a, b]$. Here, $\mathcal{T}() = d/dt$ is an operator from, $X = C^{(1)}[a, b]$, the space of continuously differentiable functions, to the space of continuous function, i.e. $Y = C[a, b]$. It is easy to check that this is a linear operator.

4. Consider transformation defined by definite integration operator, i.e.

$$\alpha = \int_a^b x(\tau)d\tau = \mathcal{T}[x(\tau)]$$

which maps $X =$ {space of integrable functions over $[a, b]$} into $Y = R$.

5. Consider ODE-IVP

$$dx/dt = f[t, x(t)], \quad t \in [0, \infty) \qquad \text{-------- (4)}$$

with initial condition $x(0) = \alpha$. Defining product space $Y = C^{(1)}[a, \infty) \times R$, the transformation $\mathcal{T}: C^{(1)}[0, \infty) \to Y$ can be stated as

$$\mathcal{T}[x(t)] = [dx/dt - f(t, x(t)), x(0)]$$

and the ODE-IVP can be represented as

$$\mathcal{T}[x(t)] = (\overline{\mathbf{0}}(t), \alpha)$$

where $\overline{\mathbf{0}}$ represents zero function over interval $[0,\infty), i.e.\ \overline{\mathbf{0}}(t) = 0$ for $t \in [0,\infty)$.

6. Consider ODE-BVP

$$a\frac{d^2u}{dz^2} + b\frac{du}{dz} + cg(u) = 0 \qquad (0 \leq z \leq 1)$$

$$B.C.\ at\, z = 0 : f_1\left[\frac{du(0)}{dz}, u(0)\right] = \alpha_0$$

$$B.C.\ at\, z = 1 : f_2\left[\frac{du(1)}{dz}, u(1)\right] = \alpha_1$$

In this case, the transformation $\mathcal{T}[u(z)]$ defined as

$$\mathcal{T}[u(z)] = \left[a\frac{d^2u(z)}{dz^2} + b\frac{du(z)}{dz} + cg(u(z)), f_1(u'(0), u(0)), f_2(u'(1), u(1))\right]$$

maps space $X = C^{(2)}[0,1]$ to $Y = C^{(2)}[0,1] \times R \times R$ and the ODE-BVP can be represented as follows

$$\mathcal{T}[u(z)] = (\overline{\mathbf{0}}(z), \alpha_0, \alpha_1)$$

7. Consider general PDE

$$a\frac{\partial^2 u}{\partial z^2} + b\frac{\partial u}{\partial z} + cg(u) - \frac{\partial u}{\partial t} = 0$$

defined over $(0 < z < 1)$ and $t \geq 0$ with the initial and the boundary conditions specified as follows

$$u(z,0) = h(z) \quad for\ (0 < z < 1)$$

$$B.C.\ at\, z = 0 : f_1\left[\frac{du(0,t)}{dz}, u(0,t)\right] = \alpha_0\ for\, t \geq 0$$

$$B.C.\ at\, z = 1 : f_2\left[\frac{du(1,t)}{dz}, u(1)\right] = \alpha_1\ \ for\, t \geq 0$$

In this case, the transformation $\mathcal{T}[u(z,t)]$ defined as

$$\mathcal{T}[u(z,t)] = a\frac{\partial^2 u(z,t)}{\partial z^2} + b\frac{\partial u(z,t)}{\partial z}$$
$$+ cg(u(z,t)) - \frac{\partial u}{\partial t}, u(z,0), f_1(u'(0,t), u(0,t)), f_2(u'(1,t), u(1,t))$$

maps space $X = C^{(2)}[0,1] \times C^{(1)}[0,\infty)$ to $Y = C^{(2)}[0,1] \times C[a,b] \times R \times R$ and the PDE can be represented as follows

$$\mathcal{T}[u(z,t)] = (\overline{\mathbf{0}}(z,t), h(z), \alpha_0, \alpha_1)$$

large number of problems arising in applied mathematics can be stated as follows [4]:

$$\text{Solve equation } \mathbf{y} = \mathcal{T}(\mathbf{x}) \qquad \text{-------- (5)}$$
$$\text{where } \mathbf{x} \in M \subset X, \mathbf{y} \in Y$$

ere, X and Y are vector spaces and operator $\mathcal{T} : M \to Y$. In engineering parlance, $\mathbf{x}, \mathbf{y}$ and $\mathcal{T}$ epresent input, output and model, respectively. Linz [4] proposes following broad classification of roblems encountered in computational mathematics

- **Direct Problems:** Given operator $\mathcal{T}$ and $\mathbf{x}$, find $\mathbf{y}$. In this case, we are trying to compute output of a given system given input. The computation of definite integrals is an example of this type.

- **Inverse Problems:** Given operator $\mathcal{T}$ and $\mathbf{y}$, find $\mathbf{x}$. In this case we are looking for input which generates the observed output. Solving system of simultaneous (linear / nonlinear) algebraic equations, ordinary and partial differential equations and integral equations are examples of this category

- **Identification problems:** Given operator $\mathbf{x}$ and $\mathbf{y}$, find $\mathcal{T}$. In this case, we try to find the laws governing systems from the knowledge of relation between the inputs and outputs.

The direct problems can be treated relatively easily. The inverse problems and the identification problems are more difficult to solve and form the central theme of this numerical analysis course. When the operator involved is nonlinear, it is difficult to solve the problem (FP-1) analytically. The problem is approximated and transformed to a computable form

$$[\mathbf{y}=\mathcal{T}(\mathbf{x})] \xrightarrow{\text{Discretization}} [\widetilde{\mathbf{y}}=\widetilde{\mathcal{T}}(\widetilde{\mathbf{x}})] \quad \text{--------} \ (6)$$

where $\widetilde{\mathbf{x}} \in X_n, \widetilde{\mathbf{y}} \in Y_n$ are finite dimensional spaces and $\widetilde{\mathcal{T}}(.)$ is an approximation of the original operator $\mathcal{T}(.)$. This process is called as *discretization.* The main strategy used for discretization is approximation of continuous functions using finite order polynomials. In the sections that follow, we discuss the theoretical basis for this choice and different commonly used polynomial based approaches for problem discretization.

Discretization using Taylor Series Approximation

1 Local approximation by Taylor series expansion

To begin with let us consider Taylor series expansion for a real valued scalar function. Given any scalar function $f(x) : R \to R,$ which is continuously differentiable $n+1$ times at $x = \overline{x}$, the Taylor series expansion of this function attempts to construct a local polynomial approximation of the form

$$p_n(x) = \alpha_0 + \alpha_1(x - \overline{x}) + \ldots\ldots + \alpha_n(x - \overline{x})^n$$

of $f(x)$ in the neighborhood of a point, say $x = \overline{x}$, such that

$$\frac{d^k p_n(\overline{x})}{dx^k} = \frac{d^k f(\overline{x})}{dx^k}$$

for k = 0,1,2,...n. For k = 0, we have

$$p_n(\overline{x}) = \alpha_0 = f(\overline{x})$$

Similarly, for k = 1, the derivative condition (dc) reduces to

$$\frac{dp_n(\overline{x})}{dx} = \left[\alpha_1 + 2\alpha_2(x - \overline{x}) + \ldots\ldots + n\alpha_n(x - \overline{x})^{n-1}\right]_{x=\overline{x}}$$

$$\Rightarrow \alpha_1 = \frac{df(\overline{x})}{dx}$$

and, in general for the k'th derivative, we have

$$\frac{d^k p_n(\overline{x})}{dx^k} = \left[(k!)\alpha_k + ((k+1)k\ldots 2)\alpha_{k+1}(x - \overline{x}) + \ldots\ldots + (n(n-1)\ldots(n-k))\alpha_n(x - \overline{x})^{n-k}\right]_{x=\overline{x}}$$

$$\Rightarrow \alpha_k = \frac{1}{k!}\frac{d^k f(\overline{x})}{dx^k}$$

Thus, the local polynomial approximation $p_n(x)$ can be expressed as

$$p_n(x) = f(\overline{x}) + \left[\frac{df(\overline{x})}{dx}\right]\delta x + \frac{1}{2!}\left[\frac{d^2 f(\overline{x})}{dx^2}\right](\delta x)^2 + \ldots + \frac{1}{n!}\left[\frac{d^n f(\overline{x})}{dx^n}\right].(\delta x)^n$$

where $\delta x = x - \overline{x}$. The residual or the approximation error, $r_n(\overline{x},\delta x)$, is defined as follows

$$r_n(\overline{x},\delta x) = f(x) - p_n(x)$$

plays an important role in analysis. The Taylor theorem gives the following analytical expression for the residual term

$$r_n(\overline{x},\delta x) = \frac{1}{(n+1)!}\frac{d^{n+1}f(\overline{x}+\lambda\delta x)}{dx^{n+1}}(\delta x)^{n+1} \quad where \quad (0 < \lambda < 1)$$

which is derived by application of the mean value theorem and the Rolle's theorem on interval $[\overline{x},x]$ PhilipsTaylor. Thus, given a scalar function $f(x) : R \to R$, which is continuously differentiable $n+1$ times at $x = \overline{x}$, the Taylor series expansion of this function can be expressed as follows

$$f(x) = f(\overline{x}) + \left[\frac{df(\overline{x})}{dx}\right]\delta x + \frac{1}{2!}\left[\frac{d^2f(\overline{x})}{dx^2}\right](\delta x)^2 + \ldots + \frac{1}{n!}\left[\frac{d^nf(\overline{x})}{dx^n}\right].(\delta x)^n + r_n(\overline{x},\delta x)$$

While developing numerical methods, we require a more general, multi-dimensional version of the Taylor series expansion. Given function $\mathbf{F}(\mathbf{x}) : R^n \to R^m$, which is continuously differentiable $n+1$ times at $\mathbf{x} = \overline{\mathbf{x}}$, the Taylor series expansion of this function in the neighborhood the point $\mathbf{x} = \overline{\mathbf{x}}$ can be expressed as follows

$$\mathbf{F}(\mathbf{x}) = \mathbf{P}_n(\mathbf{x}) + \mathbf{R}_n(\overline{\mathbf{x}},\delta\mathbf{x})$$

$$\mathbf{P}_n(\mathbf{x}) = \mathbf{F}(\overline{\mathbf{x}}) + \left[\frac{\partial\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}}\right]\delta\mathbf{x} + \frac{1}{2!}\left[\frac{\partial^2\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}^2}\right](\delta\mathbf{x},\delta\mathbf{x}) + \ldots + \frac{1}{n!}\left[\frac{\partial^n\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}^n}\right](\delta\mathbf{x},\delta\mathbf{x},\ldots,\delta\mathbf{x})$$

where $\delta\mathbf{x} = \mathbf{x} - \overline{\mathbf{x}}$ and the residual $\mathbf{R}_n(\overline{\mathbf{x}},\delta\mathbf{x})$ is defined as follows

$$\mathbf{R}_n(\overline{\mathbf{x}},\delta\mathbf{x}) = \frac{1}{(n+1)!}\frac{\partial^{n+1}\mathbf{F}(\overline{\mathbf{x}}+\lambda\delta\mathbf{x})}{\partial\mathbf{x}^{n+1}}(\delta\mathbf{x},\delta\mathbf{x},\ldots,\delta\mathbf{x}) \quad where \quad (0 < \lambda < 1)$$

Here, the $\mathbf{F}(\overline{\mathbf{x}}) \in R^m$, Jacobian $\left[\frac{\partial\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}}\right]$ is a matrix of dimension $(m \times n)$, $\left[\frac{\partial^2\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}^2}\right]$ is a $(m \times n \times n)$ dimensional array and so on. In general, $\left[\frac{\partial^r\mathbf{F}(\overline{\mathbf{x}})}{\partial\mathbf{x}^r}\right]$ is an $(m \times n \times n \ldots \times n)$ dimensional array such that when the vector $\delta\mathbf{x}$ operates on it n times, the result is an $m \times 1$ vector. It may be noted that the multi-dimensional polynomial given by equation (Pn) satisfies the condition

$$\frac{d^k\mathbf{P}_n(\overline{\mathbf{x}})}{d\mathbf{x}^k} = \frac{d^k\mathbf{F}(\overline{\mathbf{x}})}{d\mathbf{x}^k}$$

for i = 1,2,...n. The following two multidimensional cases are used very frequently in the numerical analysis.

- **Case A: Scalar Function** $f(\mathbf{x}) : R^n \to R$

$$f(\mathbf{x}) = f(\overline{\mathbf{x}}) + [\nabla f(\overline{\mathbf{x}})]^T\delta\mathbf{x} + \frac{1}{2!}\delta\mathbf{x}^T\left[\nabla^2 f(\overline{\mathbf{x}})\right]\delta\mathbf{x} + R_3(\overline{\mathbf{x}},\delta\mathbf{x})$$

$$\nabla f(\overline{\mathbf{x}}) = \left[\frac{\partial f(\overline{\mathbf{x}})}{\partial\mathbf{x}}\right] = \left[\begin{array}{cccc} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \ldots\ldots & \frac{\partial f}{\partial x_n} \end{array}\right]^T_{\mathbf{x}=\overline{\mathbf{x}}}$$

$$\nabla^2 f(\overline{\mathbf{x}}) = \left[\frac{\partial^2 f(\overline{\mathbf{x}})}{\partial \mathbf{x}^2}\right] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}=\overline{\mathbf{x}}}$$

$$R_3(\overline{\mathbf{x}}, \delta\mathbf{x}) = \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^3 f(\overline{\mathbf{x}} + \lambda\delta\mathbf{x})}{\partial x_i \partial x_j \partial x_k}\delta x_i \delta x_j \delta x_k \quad ; \quad (0 < \lambda < 1)$$

Here, $\nabla f(\overline{\mathbf{x}})$, referred to as gradient, is an $n \times 1$ vector and, $[\nabla^2 f(\overline{\mathbf{x}})]$, known as Hessian, is an $n \times n$ matrix. It may be noted that the Hessian is always a symmetric matrix.

Example 7 Consider the function vector $f(\mathbf{x}) : R^2 \rightarrow R$

$$f(\mathbf{x}) = x_1^2 + x_2^2 + e^{(x_1+x_2)}$$

which can be approximated in the neighborhood of $\overline{\mathbf{x}} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ using the Taylor series expansion

as

$$f(\mathbf{x}) = f(\overline{\mathbf{x}}) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\overline{\mathbf{x}}} \delta\mathbf{x} + \frac{1}{2}[\delta\mathbf{x}]^T \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{\mathbf{x}=\overline{\mathbf{x}}} \delta\mathbf{x} + R_3(\overline{\mathbf{x}}, \delta\mathbf{x})$$

$$= (2+e^2) + \begin{bmatrix} (2+e^2) & (2+e^2) \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^T \begin{bmatrix} (2+e^2) & e^2 \\ e^2 & (2+e^2) \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + R_3(\overline{\mathbf{x}}, \delta\mathbf{x})$$

- **Case B: Function vector** $F(\mathbf{x}) : R^n \rightarrow R^n$

$$F(\mathbf{x}) = F(\overline{\mathbf{x}}) + \left[\frac{\partial F(\overline{\mathbf{x}})}{\partial \mathbf{x}}\right]\delta\mathbf{x} + \mathbf{R}_2(\overline{\mathbf{x}}, \delta\mathbf{x})$$

$$\left[\frac{\partial F(\overline{\mathbf{x}})}{\partial \mathbf{x}}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots\cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\overline{\mathbf{x}}}$$

Here, $\left[\dfrac{\partial F(\overline{\mathbf{x}})}{\partial \mathbf{x}}\right]$, referred to as Jacobian matrix is an $n \times n$ matrix.

Example 8 Consider the function vector $F(\mathbf{x}) \in R^2$

$$F(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 + 2x_1x_2 \\ x_1x_2e^{(x_1+x_2)} \end{bmatrix}$$

which can be approximated in the neighborhood of $\overline{\mathbf{x}} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ using the Taylor series expansion as follows

$$F(\mathbf{x}) = \begin{bmatrix} f_1(\overline{\mathbf{x}}) \\ f_2(\overline{\mathbf{x}}) \end{bmatrix} + \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\overline{\mathbf{x}}} \delta\mathbf{x} + R_2(\overline{\mathbf{x}}, \delta\mathbf{x})$$

$$= \begin{bmatrix} 4 \\ e^2 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 2e^2 & 2e^2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + R_2(\overline{\mathbf{x}}, \delta\mathbf{x})$$

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