Solving Linear Algebraic Equations II

Iterative Solution Techniques

Iterative Solution Techniques

By this approach, we start with some initial *guess solution*, say $\mathbf{x}^{(0)}$, for solution $\mathbf{x}$ and generate an improved solution estimate $\mathbf{x}^{(k+1)}$ from the previous approximation $\mathbf{x}^{(k)}$. This method is a very effective for solving differential equations, integral equations and related problems Kelley. Let the residue vector $\mathbf{r}$ be defined as

$$r_i^{(k)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k)} \text{ for } i = 1, 2, \ldots n \quad \text{--------(57)}$$

i.e. $\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$. The iteration sequence $\left\{ \mathbf{x}^{(k)} : k = 0, 1, \ldots \right\}$ is terminated when some norm of the residue $\left\| \mathbf{r}^{(k)} \right\| = \left\| \mathbf{A}\mathbf{x}^{(k)} - \mathbf{b} \right\|$ becomes sufficiently small, i.e.

$$\frac{\left\| \mathbf{r}^{(k)} \right\|}{\|\mathbf{b}\|} < \varepsilon \quad \text{--------(58)}$$

where ε is an arbitrarily small number (such as 10^{-8} or 10^{-10}). Another possible termination criterion can be

$$\frac{\left\| \mathbf{x}^{(k)} - \mathbf{x}^{(k+1)} \right\|}{\left\| \mathbf{x}^{(k+1)} \right\|} < \varepsilon \quad \text{--------(59)}$$

It may be noted that the later condition is *practically* equivalent to the previous termination condition.

A simple way to form an iterative scheme is Richardson iterations Kelley

$$\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)} + \mathbf{b} \quad \text{--------(60)}$$

or Richardson iterations preconditioned with approximate inversion

$$\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{M}\mathbf{A})\mathbf{x}^{(k)} + \mathbf{M}\mathbf{b} \quad \text{--------(61)}$$

where matrix $\mathbf{M}$ is called approximate inverse of $\mathbf{A}$ if $\|\mathbf{I} - \mathbf{M}\mathbf{A}\| < 1$. A question that naturally arises is '*will the iterations converge to the solution of* $\mathbf{A}\mathbf{x} = \mathbf{b}$*?*'. In this section, to begin with, some well known iterative schemes are presented. Their convergence analysis is presented next. In the derivations that follow, it is implicitly assumed that the diagonal elements of matrix $\mathbf{A}$ are non-zero, i.e. $a_{ii} \neq 0$. If this is not the case, simple row exchange is often sufficient to satisfy this condition.

1 Iterative Algorithms
1.1 Jacobi-Method

Suppose we have a guess solution, say $\mathbf{x}^{(k)}$,

$$\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & x_2^{(k)} & \ldots. & x_n^{(k)} \end{bmatrix}^T$$

for $\mathbf{A}\mathbf{x} = \mathbf{b}$. To generate an improved estimate starting from $\mathbf{x}^{(k)}$, consider the first equation in the set of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e.,

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \quad \text{--------(62)}$$

Rearranging this equation, we can arrive at a iterative formula for computing $x_1^{(k+1)}$, as

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} \ldots\ldots - a_{1n}x_n^{(k)}\right] \quad \text{--------(63)}$$

Similarly, using second equation from $\mathbf{Ax} = \mathbf{b}$, we can derive

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} \ldots\ldots - a_{2n}x_n^{(k)}\right] \quad \text{--------(64)}$$

Table 1: Algorithm for Jacobi Iterations

INITIALIZE : $\mathbf{b}, \mathbf{A}, \mathbf{x}^{(0)}, k_{\max}, \varepsilon$

$k = 0$

$\delta = 100 * \varepsilon$

WHILE $[(\delta > \varepsilon)\ AND\ (k < k_{\max})]$

FOR $i = 1 : n$

$r_i = b_i - \sum_{j-1}^{n} a_{ij}x_j$

$x_{Ni} = x_i + \left(\frac{r_i}{a_{ii}}\right)$

END FOR

$\delta = \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$

$\times = \mathbf{x}_N$

$k = k + 1$

END WHILE

In general, using i^{th} row of $\mathbf{Ax} = \mathbf{b}$, we can generate improved guess for the i'th element $\mathbf{x}$ of as follows

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_2 - a_{i1}x_1^{(k)} \ldots\ldots - a_{i,i-1}x_{i-1}^{(k)} - a_{i,i+1}x_{i+1}^{(k)} \ldots\ldots - a_{i,n}x_n^{(k)}\right] \quad \text{--------(65)}$$

The above equation can also be rearranged as follows

$$x_i^{(k+1)} = x_i^{(k)} + \left(\frac{r_i^{(k)}}{a_{ii}}\right)$$

where $r_i^{(k)}$ is defined by equation (Ri). The algorithm for implementing the Jacobi iteration scheme is summarized in Table 1

1.2 Gauss-Seidel Method

When matrix $\mathbf{A}$ is large, there is a practical difficulty with the Jacobi method. It is required to store all components of $\mathbf{x}^{(k)}$ in the computer memory (as a separate variables) until calculations of $\mathbf{x}^{(k+1)}$ is over. The Gauss-Seidel method overcomes this difficulty by using $x_i^{(k+1)}$ immediately in the next equation while computing $x_{i+1}^{(k+1)}$. This modification leads to the following set of equations

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \ldots\ldots a_{1n}x_n^{(k)}\right] \quad \text{--------(66)}$$

Table 2: Algorithm for Gauss-Seidel Iterations

INITIALIZE : $\mathbf{b}, \mathbf{A}, \mathbf{x}, k_{\max}, \varepsilon$

$k = 0$

$\delta = 100 * \varepsilon$

WHILE $[(\delta > \varepsilon)\ AND\ (k < k_{\max})]$

FOR $i = 1 : n$

$$r_i = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$x_i = x_i + (r_i / a_{ii})$

END FOR

$\delta = \|\mathbf{r}\| / \|\mathbf{b}\|$

$k = k + 1$

END WHILE

$$x_2^{(k+1)} = \frac{1}{a_{22}} \left[b_2 - \left\{ a_{21} x_1^{(k+1)} \right\} - \left\{ a_{23} x_3^{(k)} + \ldots\ldots + a_{2n} x_n^{(k)} \right\} \right] \quad \text{--------(67)}$$

$$x_3^{(k+1)} = \frac{1}{a_{33}} \left[b_3 - \left\{ a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} \right\} - \left\{ a_{34} x_4^{(k)} + \ldots\ldots + a_{3n} x_n^{(k)} \right\} \right] \quad \text{--------(68)}$$

In general, for i'th element of $\mathbf{x}$, we have

$$x_i^{(k+1)} = \left(\frac{1}{a_{ii}} \right) \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

To simplify programming, the above equation can be rearranged as follows

$$x_i^{(k+1)} = x_i^{(k)} + \left(r_i^{(k)} / a_{ii} \right) \quad \text{--------(69)}$$

where

$$r_i^{(k)} = \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)} \right]$$

The algorithm for implementing Gauss-Siedel iteration scheme is summarized in Table 2

1.3 Relaxation Method

Suppose we have a starting value say y, of a quantity and we wish to approach a target value, say y^*, by some method. Let application of the method change the value from y to $\hat{y}$. If $\hat{y}$ is between y and $\tilde{y}$, which is even closer to y^* than $\hat{y}$, then we can approach y^* faster by magnifying the change ($\hat{y} - y$) Strang. In order to achieve this, we need to apply a magnifying factor $\omega > 1$ and get

Table 3: Algorithms for Over-Relaxation Iterations

INITIALIZE : $\mathbf{b}, \mathbf{A}, \mathbf{x}, k_{\max}, \varepsilon, \omega$

$k = 0$

$\delta = 100 * \varepsilon$

WHILE $[(\delta > \varepsilon)\ AND\ (k < k_{\max})]$

FOR $i = 1 : n$

$$r_i = b_i - \sum_{j-1}^{n} a_{ij} x_j$$

$z_i = x_i + (r_i / a_{ii})$

$x_i = x_i + \omega(z_i - x_i)$

END FOR

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$$

$\delta = \|\mathbf{r}\| / \|\mathbf{b}\|$

$k = k + 1$

END WHILE

$$\tilde{y} = y + \omega(\hat{y} - y) \quad \text{--------(70)}$$

This amplification process is an *extrapolation* and is an example of **over-relaxaation**. If the intermediate value $\hat{y}$ tends to overshoot target y^*, then we may have to use $\omega < 1$; this is called **under-relaxation**.

Application of **over-relaxation** to Gauss-Seidel method leads to the following set of equations

$$x_i^{(k+1)} = x_i^{(k)} + \omega[z_i^{(k+1)} - x_i^{(k)}] \quad \text{--------(71)}$$
$$i = 1, 2, \ldots . n$$

where $z_i^{(k+1)}$ are generated using the Gauss-Seidel method, i.e.,

$$z_i^{(k+1)} = \left(\frac{1}{a_{ii}}\right)\left[b_i - \sum_{j-1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j-i+1}^{n} a_{ij} x_j^{(k)} \right] \quad \text{--------(72)}$$
$$i = 1, 2, \ldots . n$$

The steps in the implementation of the over-relaxation iteration scheme are summarized in Table $T_O R$.

It may be noted that ω is a *tuning parameter,* which is chosen such that $1 < \omega < 2.$

\caption{Algorithms for Over-Relaxation Iterations}

2 Convergence Analysis of Iterative Methods [3, 2]

2.1 Vector-Matrix Representati

When $\mathbf{Ax} = \mathbf{b}$ is to be solved iteratively, a question that naturally arises is 'under what conditions the iterations converge?'. The convergence analysis can be carried out if the above set of iterative equations are expressed in the vector-matrix notation. For example, the iterative equations in the Gauss-Siedel method can be arranged as follows

$$\begin{bmatrix} a_{11} & 0 & .. & 0 \\ a_{21} & a_{22} & 0 & . \\ \ldots & \ldots & \ldots & 0 \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ \ldots \\ \ldots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \ldots & -a_{1,n} \\ 0 & 0 & -a_{23} & \ldots & -a_{2,n} \\ . & . & . & .. & -a_{n-1,n} \\ 0 & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ . \\ . \\ x_n^{(k)} \end{bmatrix} + \begin{bmatrix} b_1 \\ . \\ . \\ b_n \end{bmatrix} \quad (73)$$

Let $\mathbf{D}, \mathbf{L}$ and $\mathbf{U}$ be diagonal, strictly lower triangular and strictly upper triangular parts of $\mathbf{A}$, i.e.,

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \quad \text{--------(74)}$$

(*The representation given by equation (74) should NOT be confused with matrix factorization*

$\mathbf{A} = \mathbf{LDU}$). Using these matrices, the Gauss-Siedel iteration can be expressed as follows

$$(\mathbf{L} + \mathbf{D})\mathbf{x}^{(k+1)} = -\mathbf{U}\mathbf{x}^{(k)} + \mathbf{b} \quad \text{--------(75)}$$

or

$$\mathbf{x}^{(k+1)} = -(\mathbf{L} + \mathbf{D})^{-1}\mathbf{U}\mathbf{x}^{(k)} + (\mathbf{L} + \mathbf{D})^{-1}\mathbf{b} \quad \text{--------(76)}$$

Similarly, rearranging the iterative equations for Jacobi method, we arrive at

$$\mathbf{x}^{(k+1)} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b} \quad \text{--------(77)}$$

and for the relaxation method we get

$$\mathbf{x}^{(k+1)} = (\mathbf{D} + \omega\mathbf{L})^{-1}[(1-\omega)\mathbf{D} - \omega\mathbf{U}]\mathbf{x}^{(k)} + \omega\mathbf{b} \quad \text{--------(78)}$$

Thus, in general, an iterative method can be developed by splitting matrix $\mathbf{A}$. If $\mathbf{A}$ is expressed as

$$\mathbf{A} = \mathbf{S} - \mathbf{T} \quad \text{--------(79)}$$

then, equation $\mathbf{Ax} = \mathbf{b}$ can be expressed as

$$\mathbf{Sx} = \mathbf{Tx} + \mathbf{b}$$

Starting from a guess solution

$$\mathbf{x}^{(0)} = [x_1^{(0)} \ldots\ldots\ldots x_n^{(0)}]^T \quad \text{--------(80)}$$

we generate a sequence of approximate solutions as follows

$$\mathbf{x}^{(k+1)} = \mathbf{S}^{-1}[\mathbf{T}\mathbf{x}^{(k)} + \mathbf{b}] \quad where \quad k = 0, 1, 2, \ldots\ldots \quad \text{--------(81)}$$

Requirements on $\mathbf{S}$ and $\mathbf{T}$ matrices are as follows [3] : matrix A should be decomposed into $\mathbf{A} = \mathbf{S} - \mathbf{T}$ such that

- Matrix $\mathbf{S}$ should be easily invertible
- Sequence $\left\{\mathbf{x}^{(k)} : k = 0, 1, 2, \ldots\ldots\right\}$ should converge to $\mathbf{x}^*$ where $\mathbf{x}^*$ is the solution of $\mathbf{Ax} = \mathbf{b}$.

The popular iterative formulations correspond to the following choices of matrices $\mathbf{S}$ and $\mathbf{T}$ [3,4]

- **Jacobi Method:**

$$\mathbf{S}_{JAC} = \mathbf{D}, \quad \mathbf{T}_{JAC} = -(\mathbf{L} + \mathbf{U}) \quad \text{--------(82)}$$

- **Forward Gauss-Seidel Method**

$$\mathbf{S}_{GS} = \mathbf{L} + \mathbf{D}, \quad \mathbf{T}_{GS} = -\mathbf{U} \quad \text{--------(83)}$$

- **Relaxation Method:**

$$\mathbf{S}_{SOR} = \omega\mathbf{L} + \mathbf{D}, \quad \mathbf{T}_{SOR} = (1-\omega)\mathbf{D} - \omega\mathbf{U} \quad \text{--------(84)}$$

- **Backward Gauss Seidel:** In this case, iteration begins the update of $\mathbf{x}$ with n'th coordinate rather than the first. This results in the following splitting of matrix $\mathbf{A}$ [4]

$$\mathbf{S}_{BSG} = \mathbf{U} + \mathbf{D}, \quad \mathbf{T}_{BSG} = -\mathbf{L} \quad \text{--------(85)}$$

In **Symmetric Gauss Seidel** approach**,** a forward Gauss-Seidel iteration is followed by a backward Gauss-Seidel iteration.

2.2 Iterative Scheme as a Linear Difference Equation

In order to solve equation (LAE), we have formulated an iterative scheme

$$\mathbf{x}^{(k+1)} = \left(\mathbf{S}^{-1}\mathbf{T}\right)\mathbf{x}^{(k)} + \mathbf{S}^{-1}\mathbf{b} \qquad \text{--------(86)}$$

Let the true solution equation (LAE) be

$$\mathbf{x}^{*} = \left(\mathbf{S}^{-1}\mathbf{T}\right)\mathbf{x}^{*} + \mathbf{S}^{-1}\mathbf{b} \qquad \text{--------(87)}$$

Defining error vector

$$\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^{*} \qquad \text{--------(88)}$$

and subtracting equation (true) from equation (Itr), we get

$$\mathbf{e}^{(k+1)} = \left(\mathbf{S}^{-1}\mathbf{T}\right)\mathbf{e}^{(k)} \qquad \text{--------(89)}$$

Thus, if we start with some $\mathbf{e}^{(0)}$, then after k iterations we have

--------(90)

--------(91)

$$\mathbf{e}^{(1)} = \left(\mathbf{S}^{-1}\mathbf{T}\right)\mathbf{e}^{(0)}$$

$$\mathbf{e}^{(2)} = \left(\mathbf{S}^{-2}\mathbf{T}^{2}\right)\mathbf{e}^{(1)} = [\mathbf{S}^{-1}\mathbf{T}]^{2}\mathbf{e}^{(0)} \qquad \text{--------(92)}$$

$$\ldots\ldots = \ldots\ldots\ldots$$

$$\mathbf{e}^{(k)} = [\mathbf{S}^{-1}\mathbf{T}]^{k}\mathbf{e}^{(0)}$$

--------(93)

The convergence of the iterative scheme is assured if

--------(94)

$$\lim_{k \to \infty} \mathbf{e}^{(k)} = \overline{\mathbf{0}}$$

$$\text{i.e.} \quad \lim_{k \to \infty} [\mathbf{S}^{-1}\mathbf{T}]^{k}\mathbf{e}^{(0)} = \overline{\mathbf{0}}$$

--------(95)

for **any** initial guess vector $\mathbf{e}^{(0)}$.

Alternatively, consider application of the general iteration equation (86) k times starting from initial guess $\mathbf{x}^{(0)}$. At the k'th iteration step, we have

$$\mathbf{x}^{(k)} = \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k}\mathbf{x}^{(0)} + \left[\left(\mathbf{S}^{-1}\mathbf{T}\right)^{k-1} + \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k-2} + \ldots + \mathbf{S}^{-1}\mathbf{T} + \mathbf{I}\right]\mathbf{S}^{-1}\mathbf{b} \quad \text{--------(96)}$$

If we select $\left(\mathbf{S}^{-1}\mathbf{T}\right)$ such that

$$\lim_{k \to \infty} \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k} \to [\mathbf{0}] \quad \text{--------(97)}$$

where $[\mathbf{0}]$ represents the null matrix, then, using identity

$$\left[\mathbf{I} - \left(\mathbf{S}^{-1}\mathbf{T}\right)\right]^{-1} = \mathbf{I} + \left(\mathbf{S}^{-1}\mathbf{T}\right) + \ldots + \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k-1} + \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k} + \ldots$$

we can write

$$\mathbf{x}^{(k)} \to \left[\mathbf{I} - \left(\mathbf{S}^{-1}\mathbf{T}\right)\right]^{-1}\mathbf{S}^{-1}\mathbf{b} = [\mathbf{S} - \mathbf{T}]^{-1}\mathbf{b} = \mathbf{A}^{-1}\mathbf{b}$$

for large k. The above expression clearly explains how the iteration sequence generates a numerical approximation to $\mathbf{A}^{-1}\mathbf{b}$, provided condition (97) is satisfied.

2.3 Convergence Criteria for Iteration Schemes

It may be noted that equation (89) is a linear difference equation of form

$$\mathbf{z}^{(k+1)} = \mathbf{B}\mathbf{z}^{(k)} \quad \text{--------(98)}$$

with a specified initial condition $\mathbf{z}^{(0)}$. Here, $\mathbf{z} \in \mathbf{R}^{n}$ and $\mathbf{B}$ is a $n \times n$ matrix. In Appendix A, we analyzed behavior of the solutions of linear difference equations of type (98). The criterion for convergence of iteration equation (89) can be derived using results derived in Appendix A. The necessary and sufficient condition for convergence of (89) can be stated as

$$\rho(\mathbf{S}^{-1}\mathbf{T}) < 1$$

i.e. the spectral radius of matrix $\mathbf{S}^{-1}\mathbf{T}$ should be less than one.

The necessary and sufficient condition for convergence stated above requires computation of eigenvalues of $\mathbf{S}^{-1}\mathbf{T}$, which is a computationally demanding task when the matrix dimension is large. For a large dimensional matrix, if we could check this condition before starting the iterations, then we might as well solve the problem by a direct method rather than using iterative approach to save computations. Thus, there is a need to derive some alternate criteria for convergence, which can be checked easily before starting iterations. Theorem 14 in Appendix A states that spectral radium of a matrix is smaller than any induced norm of the matrix. Thus, for matrix $\mathbf{S}^{-1}\mathbf{T}$, we have

$$\rho\left(\mathbf{S}^{-1}\mathbf{T}\right) \leq \left\|\mathbf{S}^{-1}\mathbf{T}\right\|$$

$\|.\|$ is any induced matrix norm. Using this result, we can arrive at the following sufficient conditions for the convergence of iterations

$$\left\|\mathbf{S}^{-1}\mathbf{T}\right\|_{1} < 1 \quad \text{or} \quad \left\|\mathbf{S}^{-1}\mathbf{T}\right\|_{\infty} < 1$$

Evaluating 1 or ∞ norms of $\mathbf{S}^{-1}\mathbf{T}$ is significantly easier than evaluating the spectral radius of $\mathbf{S}^{-1}\mathbf{T}$. Satisfaction of any of the above conditions implies $\rho\left(\mathbf{S}^{-1}\mathbf{T}\right) < 1$. However, it may be noted that these are only sufficient conditions. Thus, if $\left\|\mathbf{S}^{-1}\mathbf{T}\right\|_{\infty} > 1$ or $\left\|\mathbf{S}^{-1}\mathbf{T}\right\|_{1} > 1$, we cannot conclude anything about the convergence of iterations.

If the matrix $\mathbf{A}$ has some special properties, such as diagonal dominance or symmetry and positive definiteness, then the convergence is ensured for some iterative techniques. Some of the important convergence results available in the literature are summarized here.

Definition 1 :A matrix $\mathbf{A}$ is called strictly diagonally dominant if

$$\sum_{j=1(j\neq i)}^{n} |a_{ij}| < |a_{ii}| \quad for\ i = 1,2.,\ldots n \quad \text{--------(99)}$$

Theorem 2 ***[2]*** A sufficient condition for the convergence of Jacobi and Gauss-Seidel methods is that the matrix $\mathbf{A}$ of linear system $\mathbf{Ax} = \mathbf{b}$ is strictly diagonally dominant.

Proof: Refer to Appendix B.

Theorem 3 ***[5]*** The Gauss-Seidel iterations converge if matrix $\mathbf{A}$ is symmetric and positive definite.

Proof: Refer to Appendix B.

Theorem 4 ***[3]*** For an arbitrary matrix $\mathbf{A}$, the necessary condition for the convergence of relaxation method is $0 < \omega < 2.$

Proof: Refer to appendix B.

Theorem 5 ***[2]*** When matrix $\mathbf{A}$ is strictly diagonally dominant, a sufficient condition for the convergence of relaxation methods is that $0 < \omega \leq 1.$

Proof: Left to reader as an exercise.

Theorem 6 ***[2]*** For a symmetric and positive definite matrix $\mathbf{A}$, the relaxation method converges if and only if $0 < \omega < 2.$

Proof: Left to reader as an exercise.

The Theorems 3 and 6 guarantees convergence of Gauss-Seidel method or relaxation method when matrix $\mathbf{A}$ is symmetric and positive definite. Now, what do we do if matrix $\mathbf{A}$ in $\mathbf{Ax} = \mathbf{b}$ is not symmetric and positive definite? We can multiply both the sides of the equation by $\mathbf{A}^T$ and transform the original problem as follows

$$\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} = \left(\mathbf{A}^T\mathbf{b}\right) \quad \text{--------(100)}$$

If matrix $\mathbf{A}$ is non-singular, then matrix $\left(\mathbf{A}^T\mathbf{A}\right)$ is always symmetric and positive definite as

$$\mathbf{x}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} = (\mathbf{Ax})^T(\mathbf{Ax}) > 0 \text{ for any } \mathbf{x} \neq \bar{0} \quad \text{--------(101)}$$

Now, for the transformed problem, we are guaranteed convergence if we use the Gauss-Seidel method or relaxation method such that $0 < \omega < 2$.

Example 7 ***[3]*** Consider system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{--------(102)}$$

For Jacobi method

$$\mathbf{S}^{-1}\mathbf{T} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \quad \text{--------(103)}$$

$$\rho(\mathbf{S}^{-1}\mathbf{T}) = 1/2 \quad \text{--------(104)}$$

Thus, the error norm at each iteration is reduced by factor of 0.5For Gauss-Seidel method

--------(105)

$$\mathbf{S}^{-1}\mathbf{T} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$$ --------(106)

$$\rho(\mathbf{S}^{-1}\mathbf{T}) = 1/4$$

Thus, the error norm at each iteration is reduced by factor of 1/4. This implies that, for the example under consideration

$$1\ Gauss\ Seidel\ iteration\ =\ 2\ Jacobi\ iterations$$ --------(107)

For relaxation method,

--------(108)

$$\mathbf{S}^{-1}\mathbf{T} = \begin{bmatrix} 2 & 0 \\ -\omega & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix}$$ --------(109)

$$= \begin{bmatrix} (1-\omega) & (\omega/2) \\ (\omega/2)(1-\omega) & (1-\omega+\frac{\omega^2}{4}) \end{bmatrix}$$

$$\lambda_1\lambda_2 = \det(\mathbf{S}^{-1}\mathbf{T}) = (1-\omega)^2$$ --------(110)

$$\lambda_1 + \lambda_2 = trace(\mathbf{S}^{-1}\mathbf{T})$$

$$= 2 - 2\omega + \frac{\omega^2}{4}$$ --------(111)

--------(112)

Now, if we plot $\rho(\mathbf{S}^{-1}\mathbf{T})$ v/s ω, then it is observed that $\lambda_1 = \lambda_2$ at $\omega = \omega_{opt}$. From equation (110), it follows that

$$\lambda_1 = \lambda_2 = \omega_{opt} - 1$$ --------(113)

at optimum ω. Now,

--------(114)

$$\lambda_1 + \lambda_2 = 2(\omega_{opt} - 1)$$

$$= 2 - 2\omega_{opt} + \frac{\omega_{opt}^2}{4}$$ --------(115)

--------(116)

$$\Rightarrow \omega_{opt} = 4(2 - \sqrt{3}) \cong 1.07$$

$$\Rightarrow \rho(\mathbf{S}^{-1}\mathbf{T}) = \lambda_1 = \lambda_2 \cong 0.07$$ --------(117)

This is a major reduction in spectral radius when compared to Gauss-Seidel method. Thus, the error

norm at each iteration is reduced by factor of 1/16 ($\cong 0.07$) if we choose $\omega = \omega_{opt}$.

Example 8 Consider system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 4 & 5 & 9 \\ 7 & 1 & 6 \\ 5 & 2 & 9 \end{bmatrix} ; \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{--------(118)}$$

If we use Gauss-Seidel method to solve for $\mathbf{x}$, the iterations do not converge as

--------(119)

$$\mathbf{S}^{-1}\mathbf{T} = \begin{bmatrix} 4 & 0 & 0 \\ 7 & 1 & 0 \\ 5 & 2 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -5 & -9 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{--------(120)}$$

$$\rho(\mathbf{S}^{-1}\mathbf{T}) = 7.3 > 1$$

Now, let us modify the problem by pre-multiplying $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^T$ on both the sides, i.e. the modified problem is $(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{A}^T\mathbf{b})$. The modified problem becomes

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 90 & 37 & 123 \\ 37 & 30 & 69 \\ 123 & 69 & 198 \end{bmatrix}; \ \mathbf{A}^T\mathbf{b} = \begin{bmatrix} 16 \\ 8 \\ 24 \end{bmatrix} \quad \text{--------(121)}$$

The matrix $\mathbf{A}^T\mathbf{A}$ is symmetric and positive definite and, according to Theorem 3, the iterations should converge if Gauss-Seidel method is used. For the transformed problem, we have

--------(122)

$$\mathbf{S}^{-1}\mathbf{T} = \begin{bmatrix} 90 & 0 & 0 \\ 37 & 30 & 0 \\ 123 & 69 & 198 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -37 & -123 \\ 0 & 0 & -69 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{--------(123)}$$

$$\rho(\mathbf{S}^{-1}\mathbf{T}) = 0.96 < 1$$

and within 220 iterations (termination criterion 1×10^{-5}), we get following solution

$$\mathbf{x} = \begin{bmatrix} 0.0937 \\ 0.0312 \\ 0.0521 \end{bmatrix} \quad \text{--------(124)}$$

which is close to the solution

$$\mathbf{x}^* = \begin{bmatrix} 0.0937 \\ 0.0313 \\ 0.0521 \end{bmatrix} \quad \text{--------(125)}$$

computed as $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{b}$.

Table 4: Rate of Convergence of Iterative Methods

Method	Rate of Convergence	No. of iterations for ε error
Jacobi	$O(1/2n^2)$	$O(2n^2)$
Gauss_Seidel	$O(1/n^2)$	$O(n^2)$
Relaxation with optimal ω	$O(2/n)$	$O(n/2)$

Example 9 Consider system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 7 & 1 & -2 & 1 \\ 1 & 8 & 1 & 0 \\ -2 & 1 & 5 & -1 \\ 1 & 0 & -1 & 3 \end{bmatrix} ; \ \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{--------(126)}$$

If it is desired to solve the resulting problem using Jacobi method / Gauss-Seidel method, will the iterations converge? To establish convergence of Jacobi / Gauss-Seidel method, we can check whether A is strictly diagonally dominant. Since the following inequalities hold

$$Row\ 1: \ 1 + |-2| + 1 < 7$$

$$Row\ 2: \ 1 + 0 + 1 < 8$$

$$Row\ 3: \ |-2| + 1 + |-1| < 5$$

$$Row\ 4: \ 1 + 0 + |-1| < 3$$

matrix A is strictly diagonally dominant, which is a sufficient condition for convergence of Jacobi / Gauss-Seidel iterations Theorem 2. Thus, Jacobi / Gauss-Seidel iterations will converge to the solution starting from **any** initial guess.

From these examples, we can clearly see that the rate of convergence depends on $\rho(\mathbf{S}^{-1}\mathbf{T})$.

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