Solving Ordinary Differential Equations – Initial Value Problems I

WHAT IS A DIFFERENTIAL EQUATION?

An equation of the form

$$\frac{dy}{dx} = f(x)$$

that has a derivative in it is called a differential equation. Differential equations are an important topic in calculus, engineering, and the sciences. A lot of the equations that you work with in science and engineering are derived from a specific type of differential equation called an initial value problem.

INITIAL VALUE PROBLEM

The problem of finding a function y of x when we know its derivative and its value y_0 at a particular point x_0 is called an initial value problem. This problem can be solved in two steps.

1. $\int dy = \int f(x)\,dx \;\rightarrow\; y = F(x) + c \;\leftarrow$ general solution

2. Using the initial data, plug it into the general solution and solve for c.

EXAMPLE : Solve the initial value problem.

$$\frac{dy}{dx} = 10 - x, \qquad y(0) = -1$$

SOLUTION:

STEP 1:

$$\frac{dy}{dx} = 10 - x \;\rightarrow\; dy = (10 - x)\,dx$$

$$\int dy = \int (10 - x)\,dx \;\rightarrow\; y = 10x - \frac{x^2}{2} + c$$

STEP 2: When x = 0, y = -1.

$$-1 = 10(0) - \frac{0}{2} + c \;\rightarrow\; c = -1$$

$$\text{SOLUTION: } y = 10x - \frac{x^2}{2} - 1$$

Graphical and Numerical Methods

In studying the first-order ODE

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

the main emphasis is on learning different ways of finding explicit solutions. But you should realize that most first-order equations cannot be solved explicitly. For such equations, one resorts to graphical and numerical methods. Carried out by hand, the graphical methods give rough qualitative information about how the graphs of solutions to (1) look geometrically. The numerical methods then give the actual graphs to as great an accuracy as desired; the computer does the numerical work, and plots the solutions.

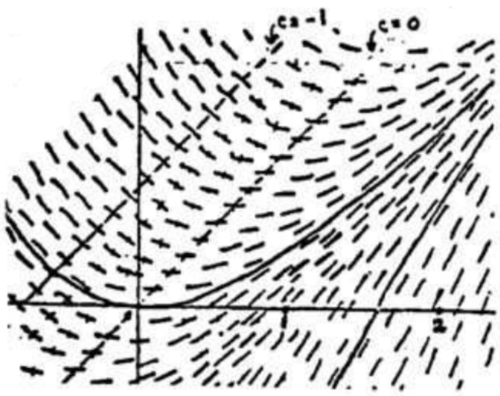
Graphical methods.

The graphical methods are based on the construction of what is called a **direction field** for the equation (1). To get this, we imagine that through each point (x, y) of the plane is drawn a little line segment whose slope is $f(x, y)$. In practice, the segments are drawn in at a representative set of points in the plane; if the computer draws them, the points are evenly spaced in both directions, forming a lattice. If drawn by hand, however, they are not, because a different procedure is used, better adapted to people.

To construct a direction field by hand, draw in lightly, or in dashed lines, what are called the **isoclines** for the equation (1). These are the one-parameter family of curves given by the equations

$$f(x,y) = c, \qquad c \text{ constant.} \tag{2}$$

Along the isocline given by the equation (2), the line segments all have the same slope c; this makes it easy to draw in those line segments, and you can put in as many as you want. (Note: "iso-cline" = "equal slope".)



The picture shows a direction field for the equation

$$y' = x - y\ .$$

The isoclines are the lines $x - y = c$, two of which are shown in dashed lines, corresponding to the values $c = 0, -1$. (Use dashed lines for isoclines).

Once you have sketched the direction field for the equation (1) by drawing some isoclines and drawing in little line segments along each of them, the next step is to draw in curves which are at each point tangent to the line segment at that point. Such curves are called **integral curves** or *solution curves* for the direction field. Their significance is this:

(3) *The integral curves are the graphs of the solutions to* $y' = f(x,y)$.

Proof. Suppose the integral curve C is represented near the point (x, y) by the graph of the function $y = y(x)$. To say that C is an integral curve is the same as saying

$$\text{slope of } C \text{ at } (x,y) \quad = \quad \text{slope of the direction field at } (x,y);$$

from the way the direction field is defined, this is the same as saying

$$y'(x) \quad = \quad f(x,y)\ .$$

But this last equation exactly says that $y(x)$ is a solution to (1).

We may summarize things by saying, the direction field gives a picture of the first-order equation (1), and its integral curves give a picture of the solutions to (1).

Two integral curves (in solid lines) have been drawn for the equation $y' = x - y$. In general, by sketching in a few integral curves, one can often get some feeling for the behavior of the solutions. The problems will illustrate. Even when the equation can be solved exactly, sometimes you learn more about the solutions by sketching a direction field and some integral curves, than by putting numerical values into exact solutions and plotting them.

There is a theorem about the integral curves which often helps in sketching them.

Integral Curve Theorem.

(i) If $f(x,y)$ is defined in a region of the xy-plane, then integral curves of $y' = f(x,y)$ cannot cross at a positive angle anywhere in that region.

(ii) If $f_y(x,y)$ is continuous in the region, then integral curves cannot even be tangent in that region.

A convenient summary of both statements is (here "smooth"= continuously differentiable):

Intersection Principle

(4) *Integral curves of $y' = f(x,y)$ cannot intersect wherever $f(x,y)$ is smooth.*

Proof of the Theorem. The first statement (i) is easy, for at any point (x_0, y_0) where they crossed, the two integral curves would have to have the same slope, namely $f(x_0, y_0)$. So they cannot cross at a positive angle.

The second statement (ii) is a consequence of the uniqueness theorem for first-order ODE's; it will be taken up then when we study that theorem. Essentially, the hypothesis guarantees that through each point (x_0, y_0) of the region, there is a unique solution to the ODE, which means there is a unique integral curve through that point. So two integral curves cannot intersect — in particular, they cannot be tangent — at any point where $f(x,y)$ has continuous derivatives.

The ODE of a family. Orthogonal trajectories.

The solution to the ODE (1) is given analytically by an xy-equation containing an arbitrary constant c; either in the explicit form (5a), or the implicit form (5b):

(5) $\qquad (a) \quad y = g(x,c) \qquad\qquad (b) \quad h(x,y,c) = 0\ .$

In either form, as the parameter c takes on different numerical values, the corresponding graphs of the equations form a one-parameter family of curves in the xy-plane.

We now want to consider the inverse problem. Starting with an ODE, we got a one-parameter family of curves as its integral curves. Suppose instead we start with a one-parameter family of curves defined by an equation of the form (5a) or (5b), can we find a first-order ODE having these as its integral curves, i.e. the equations (5) as its solutions?

The answer is yes; the ODE is found by differentiating the equation of the family (5) (using implicit differentiation if it has the form (5b)), and then using (5) to eliminate the arbitrary constant c from the differentiated equation.

Example 1. Find a first-order ODE whose general solution is the family

$$y = \frac{c}{x-c} \qquad (c \text{ is an arbitrary constant}). \tag{6}$$

Solution. We differentiate both sides of (6) with respect to x, getting $y' = -\dfrac{c}{(x-c)^2}$. We eliminate c from this equation, in steps. By (6), $x - c = c/y$, so that

$$y' = -\frac{c}{(x-c)^2} = -\frac{c}{(c/y)^2} = -\frac{y^2}{c}\ ; \tag{7}$$

To get rid of c, we solve (6) algebraically for c, getting $c = \dfrac{yx}{y+1}$; substitute this for the c on the right side of (7), then cancel a y from the top and bottom; you get as the ODE having the solution (6)

$$y' = -\frac{y(y+1)}{x}\ . \tag{8}$$

Remark. The c must not appear in the ODE, since then we would not have a single ODE, but rather a one-parameter family of ODE's — one for each possible value of c. Instead, we want just one ODE which has each of the curves (5) as an integral curve, regardless of the value of c for that curve; thus the ODE cannot itself contain c.

Orthogonal trajectories.

Given a one-parameter family of plane curves, its **orthogonal trajectories** are another one-parameter family of curves, each one of which is perpendicular to all the curves in the original family. For instance, if the original family consisted of all circles having center at the origin, its orthogonal trajectories would be all rays (half-lines) starting at the origin.

Orthogonal trajectories arise in different contexts in applications. For example, if the original family represents the **lines of force** in a gravitational or electrostatic field, its orthogonal trajectories represent the **equipotentials**, the curves along which the gravitational or electrostatic potential is constant.

More generally, if the original family is of the form $h(x, y) = c$, it represents the **level curves** of the function $h(x, y)$; its orthogonal trajectories will then be the **gradient curves** for this function — curves which everywhere have the direction of the gradient vector ∇h. This follows from the theorem which says the gradient ∇h at any point (x, y) is perpendicular to the level curve of $h(x, y)$ passing through that point.

To find the orthogonal trajectories for a one-parameter family (5):

1. Find the ODE $y' = f(x, y)$ satisfied by the family.
2. The new ODE $y' = -\dfrac{1}{f((x, y)}$ will have as its integral curves the orthogonal trajectories to the family (5); solve it to find the equation of these curves.

The method works because at any point (x, y), the orthogonal trajectory passing through (x, y) is perpendicular to the curve of the family (5a) passing through (x, y). Therefore the slopes of the two curves are negative reciprocals of each other. Since the slope of the original curve at (x, y) is $f(x, y)$, the slope at (x, y) of the orthogonal trajectory has to be $-1/f(x, y)$. The ODE for the orthogonal trajectories then gives their slope at (x, y), thus it is

$$(9) \qquad y' = -\frac{1}{f(x, y)} \qquad \textit{ODE for orthogonal trajectories to (5a)}\ .$$

More generally, if the equation of the original family is given implicitly by (5b), and its ODE is also in implicit form, the procedure and its justification are essentially the same:
1. Find the ODE in implicit form $F(x, y, y') = 0$ satisfied by the family (5).
2. Replace y' by $-1/y'$; solve the new ODE $F(x, y, -1/y') = 0$ to find the orthogonal trajectores of the original family.

Example 2. Find the orthogonal trajectories to the family of curves $y = c\,x^n$, where n is a fixed positive integer and c an arbitrary constant.

Solution. If $n = 1$, the curves are the family of rays from the origin, so the orthogonal trajectories should be the circles centered at the origin – this will help check our work.
We first find the ODE of the family. Differentiating the equation of the family gives $y' = nc\,x^{n-1}$; we eliminate c by using the equation of the family to get $c = y/x^n$ and substituting this into the differentiated equation, giving

$$(10) \qquad y' = \frac{n\,y}{x} \quad (\textit{ODE of family}); \qquad\qquad y' = -\frac{x}{n\,y} \quad (\textit{ODE of orthog. trajs.})\ .$$

Solving the latter equation by separation of variables leads first to $n\,ydy = -xdx$, then after integrating both sides, transposing, and multiplying through by 2, to the solution

$$(11) \qquad x^2 + n\,y^2 = k, \qquad (k \geq 0 \textit{ is an arbitrary non-negative constant; } n \textit{ is fixed.})$$

For different k-values, the equations (11) represent the family of ellipses centered at the origin, and having x-intercepts at $\pm\sqrt{k}$ and y-intercepts at $\pm\sqrt{k/n}$.
If $n = 1$, these intercepts are equal, and the ellipses are circles centered at the origin, as predicted.

Euler's numerical method. The graphical method gives you a quick feel for how the integral curves behave. But when they must be known accurately and the equation cannot be solved exactly, numerical methods are used. The simplest method is called **Euler's method**. Here is its geometric description.

We want to calculate the solution (integral curve) to $y' = f(x, y)$ passing through (x_0, y_0). It is shown as a curve in the picture.

We choose a step size h. Starting at (x_0, y_0), over the interval $[x_0, x_0 + h]$, we approximate the integral curve by the tangent line: the line having slope $f(x_0, y_0)$. (This is the slope of the integral curve, since $y' = f(x, y)$.)

This takes us as far as the point (x_1, y_1), which is calculated by the equations

$$x_1 = x_0 + h, \qquad y_1 = y_0 + h\, f(x_0, y_0)\ .$$

Now we are at (x_1, y_1). We repeat the process, using as the new approximation to the integral curve the line segment having slope $f(x_1, y_1)$. This takes us as far as the next point (x_2, y_2), where

$$x_2 = x_1 + h, \qquad y_2 = y_1 + h\, f(x_1, y_1)\ .$$

We continue in the same way. The general formulas telling us how to get from the $(n-1)$-st point to the n-th point are

$$x_n = x_{n-1} + h, \qquad y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})\ . \tag{12}$$

In this way, we get an approximation to the integral curve consisting of line segments joining the points (x_0, y_0), $(x_1, y_1), \ldots$.

In doing a few steps of Euler's method by hand, as you are asked to do in some of the exercises to get a feel for the method, it's best to arrange the work systematically in a table.

Example 3. For the IVP: $y' = x^2 - y^2$, $y(1) = 0$, use Euler's method with step size .1 to find y(1.2).

Solution. We use $f(x, y) = x^2 - y^2$, $h = .1$, and (12) above to find x_n and y_n:

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$
0	1	0	1	.1
1	1.1	.1	1.20	.12
2	1.2	.22		

Remarks. Euler's method becomes more accurate the smaller the step-size h is taken. But if h is too small, round-off errors can appear, particularly on a pocket calculator.

As the picture suggests, the errors in Euler's method will accumulate if the integral curve is convex (concave up) or concave (concave down). Refinements of Euler's method are aimed at using as the slope for the line segment at (x_n, y_n) a value which will correct for the convexity or concavity, and thus make the next point (x_{n+1}, y_{n+1}) closer to the true integral curve.

1. Complex arithmetic.

Most people think that complex numbers arose from attempts to solve quadratic equations, but actually it was in connection with cubic equations they first appeared. Everyone knew that certain quadratic equations, like

$$x^2 + 1 = 0, \qquad \text{or} \qquad x^2 + 2x + 5 = 0,$$

had no solutions. The problem was with certain cubic equations, for example

$$x^3 - 6x + 2 = 0.$$

This equation was known to have three real roots, given by simple combinations of the expressions

(1) $$A = \sqrt[3]{-1 + \sqrt{-7}}, \qquad B = \sqrt[3]{-1 - \sqrt{-7}};$$

one of the roots for instance is $A + B$: it may not look like a real number, but it turns out to be one.

What was to be made of the expressions A and B? They were viewed as some sort of "imaginary numbers" which had no meaning in themselves, but which were useful as intermediate steps in calculations that would ultimately lead to the real numbers you were looking for (such as $A + B$).

To describe the complex numbers, we use a formal symbol i representing $\sqrt{-1}$; then a **complex number** is an expression of the form

(2) $$a + bi, \qquad a, b \quad \text{real numbers.}$$

If $a = 0$ or $b = 0$, they are omitted (unless both are 0); thus we write

$$a + 0i = a, \qquad 0 + bi = bi, \qquad 0 + 0i = 0\ .$$

The definition of *equality* between two complex numbers is

(3) $$a + bi = c + di \quad \Leftrightarrow \quad a = c,\ b = d\ .$$

This shows that the numbers a and b are uniquely determined once the complex number $a+bi$ is given; we call them respectively the **real** and **imaginary** parts of $a+bi$. (It would be more logical to call bi the imaginary part, but this would be less convenient.) In symbols,

(4) $$a = \operatorname{Re}(a + bi), \qquad b = \operatorname{Im}(a + bi)$$

Addition and multiplication of complex numbers are defined in the familiar way, making use of the fact that $i^2 = -1$:

(5a) **Addition** $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(5b) **Multiplication** $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Division is a little more complicated; what is important is not so much the final formula but rather the procedure which produces it; assuming $c + di \neq 0$, it is:

(5c) **Division** $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i$$

This division procedure made use of *complex conjugation*: if $z = a + bi$, we define the **complex conjugate** of z to be the complex number

$$(6) \qquad \bar{z} = a - bi \qquad \text{(note that } z\bar{z} = a^2 + b^2 \text{).}$$

The size of a complex number is measured by its **absolute value**, or *modulus*, defined by

$$(7) \qquad |z| = |a + bi| = \sqrt{a^2 + b^2}; \qquad \text{(thus : } z\bar{z} = |z|^2 \text{).}$$

Remarks. For the sake of computers, which do not understand what a "formal expression" is, one can define a complex number to be just an ordered pair (a, b) of real numbers, and define the arithmetic operations accordingly; using (5b), multiplication is defined by

$$(a, b)(c, d) \ = \ (ac - bd, \ ad + bc) \ .$$

Then if we let i represent the ordered pair $(0, 1)$, and a the ordered pair $(a, 0)$, it is easy to verify using the above definition of multiplication that

$$i^2 = (0,1)(0,1) = (-1,0) = -1 \qquad \text{and} \qquad (a,b) = (a,0) + (b,0)(0,1) = a + bi \ ,$$

and we recover the human way of writing complex numbers.

Since it is easily verified from the definition that multiplication of complex numbers is commutative: $z_1 z_2 = z_2 z_1$, it does not matter whether the i comes before or after, i.e., whether we write $z = x + yi$ or $z = x + iy$. The former is used when x and y are simple numbers because it looks better; the latter is more usual when x and y represent functions (or values of functions), to make the i stand out clearly or to avoid having to use parentheses:

$$2 + 3i, \quad 5 - 2\pi i; \qquad \cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2}, \quad x(t) + i\,y(t) \ .$$

2. Polar representation.

Complex numbers are represented geometrically by points in the plane: the number $a + ib$ is represented by the point (a, b) in Cartesian coordinates. When the points of the plane represent complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we get the **polar form** for a non-zero complex number: assuming $x + iy \neq 0$,

$$(8) \qquad x + iy \ = \ r(\cos\theta + i\sin\theta) \ .$$

When the complex number is written in polar form, we see from (7) that

$$r = |x + iy|. \qquad \textit{(absolute value, modulus)}$$

We call θ the *polar angle* or the *argument* of $x + iy$. In symbols, one sometimes sees

$$\theta = \arg\,(x + iy) \qquad \textit{(polar angle, argument)}\ .$$

The absolute value is uniquely determined by $x+iy$, but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \le \theta < 2\pi \qquad \textit{principal value} \text{ of the polar angle.}$$

This so-called principal value of the angle is sometimes indicated by writing $\mathrm{Arg}\,(x+iy)$. For example,

$$\mathrm{Arg}\,(-1) = \pi, \qquad \arg\,(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots\ .$$

Changing between Cartesian and polar representation of a complex number is essentially the same as changing between Cartesian and polar coordinates: the same equations are used.

Example 1. Give the polar form for: $-i, \quad 1+i, \quad 1-i, \quad -1+i\sqrt{3}$.

Solution.

$$\begin{aligned} -i &= i\,\sin\tfrac{3\pi}{2} & 1+i &= \sqrt{2}\,(\cos\tfrac{\pi}{4} + i\,\sin\tfrac{\pi}{4}) \\ -1+i\sqrt{3} &= 2\,(\cos\tfrac{2\pi}{3} + i\,\sin\tfrac{2\pi}{3}) & 1-i &= \sqrt{2}\,(\cos\tfrac{-\pi}{4} + i\,\sin\tfrac{-\pi}{4}) \end{aligned}$$

The abbreviation $\mathrm{cis}\,\theta$ is sometimes used for $\cos\theta + i\sin\theta$; for students of science and engineering, however, it is important to get used to the exponential form for this expression:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{Euler's formula}. \tag{9}$$

Equation (9) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it however is found in the infinite series

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ .$$

If we substitute $i\theta$ for t in the series, and collect the real and imaginary parts of the sum (remembering that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ldots\ ,$$

and so on, we get

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\ &= \cos\theta \;+\; i\sin\theta\ , \end{aligned}$$

in view of the infinite series representations for $\cos\theta$ and $\sin\theta$.

> Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (9). What it shows is that Euler's formula (9) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

Using the complex exponential, the polar representation (8) is written

$$x + iy = r\,e^{i\theta} \tag{10}$$

The most important reason for polar representation is that multiplication and division of complex numbers is particularly simple when they are written in polar form. Indeed, by using Euler's formula (9) and the trigonometric addition formulas, it is not hard to show

(11) $$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}\ .$$

This gives another justification for the definition (9) — it makes the complex exponential follow the same exponential addition law as the real exponential. The law (11) leads to the simple rules for multiplying and dividing complex numbers written in polar form:

(12a) **multiplication rule** $$r\,e^{i\theta}\cdot r'e^{i\theta'} = r\,r'\,e^{i(\theta+\theta')}\ ;$$

to multiply two complex numbers, you multiply the absolute values and add the angles.

(12b) **reciprocal rule** $$\frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta};$$

(12c) **division rule** $$\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}\,e^{i\,(\theta-\theta')};$$

to divide by a complex number, divide by its absolute value and subtract its angle.

The reciprocal rule (12b) follows from (12a), which shows that $\frac{1}{r}e^{-i\theta}\cdot re^{i\theta} = 1$.

The division rule follows by writing $\frac{r\,e^{i\theta}}{r'e^{i\theta'}} = \frac{1}{r'e^{i\theta'}}\cdot r\,e^{i\theta}$ and using (12b) and then (12a).

Using (12a), we can raise $x+iy$ to a positive integer power by first using $x+iy = r\,e^{i\theta}$; the special case when $r = 1$ is called *DeMoivre's formula*:

(13) $(x+iy)^n = r^n e^{in\theta}$; **DeMoivre's formula:** $(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$.

Example 2. Express a) $(1+i)^6$ in Cartesian form; b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

Solution. a) Change to polar form, use (13), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}\,e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i\,6\pi/4} = 8\,e^{i\,3\pi/2} = -8i\ .$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (12c).

You can check the answer to (a) by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts; to (b) by doing the division in Cartesian form (5c), then converting the answer to polar form.

3. Complex exponentials

Because of the importance of complex exponentials in differential equations, and in science and engineering generally, we go a little further with them.

Euler's formula (9) defines the exponential to a pure imaginary power. The definition of an exponential to an arbitrary complex power is:

(14) $$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b).$$

We stress that the equation (14) is a definition, not a self-evident truth, since up to now no meaning has been assigned to the left-hand side. From (14) we see that

(15) $$\operatorname{Re}(e^{a+ib}) = e^a\cos b, \qquad \operatorname{Im}(e^{a+ib}) = e^a\sin b\ .$$

The complex exponential obeys the usual law of exponents:

(16) $$e^{z+z'} = e^z e^{z'} ,$$

as is easily seen by combining (14) and (11).

The complex exponential is expressed in terms of the sine and cosine by Euler's formula (9). Conversely, the sin and cos functions can be expressed in terms of complex exponentials. There are two important ways of doing this, both of which you should learn:

(17) $$\cos x = \mathrm{Re}\,(e^{ix}), \qquad \sin x = \mathrm{Im}\,(e^{ix}) ;$$

(18) $$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}) .$$

The equations in (18) follow easily from Euler's formula (9); their derivation is left for the exercises. Here are some examples of their use.

Example 3. Express $\cos^3 x$ in terms of the functions $\cos nx$, for suitable n.

Solution. We use (18) and the binomial theorem, then (18) again:

$$\begin{aligned}\cos^3 x &= \tfrac{1}{8}(e^{ix} + e^{-ix})^3 \\ &= \tfrac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x . \qquad \square\end{aligned}$$

As a preliminary to the next example, we note that a function like

$$e^{ix} = \cos x + i \sin x$$

is a *complex-valued function of the real variable* x. Such a function may be written as

$$u(x) + i\,v(x), \qquad u,\ v \text{ real-valued}$$

and its derivative and integral with respect to x are defined to be

(19a,b) $$a)\ D(u+iv) = Du + iDv, \qquad b)\ \int (u+iv)\,dx = \int u\,dx + i\int v\,dx .$$

From this it follows by a calculation that

(20) $$D(e^{(a+ib)x} = (a+ib)e^{(a+ib)x}, \qquad \text{and therefore} \qquad \int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} .$$

Example 4. Calculate $\int e^x \cos 2x\,dx$ by using complex exponentials.

Solution. The usual method is a tricky use of two successive integration by parts. Using complex exponentials instead, the calculation is straightforward. We have

$$e^x \cos 2x = \mathrm{Re}\left(e^{(1+2i)x}\right), \qquad \text{by (14) or (15); therefore}$$

$$\int e^x \cos 2x\,dx = \mathrm{Re}\left(\int e^{(1+2i)x}\,dx\right), \qquad \text{by (19b).}$$

Calculating the integral,

$$\begin{aligned}\int e^{(1+2i)x}\,dx &= \frac{1}{1+2i}\,e^{(1+2i)x} \qquad \text{by (20);} \\ &= \left(\frac{1}{5} - \frac{2}{5}i\right)\left(e^x \cos 2x + i\,e^x \sin 2x\right),\end{aligned}$$

using (14) and complex division (5c). According to the second line above, we want the real part of this last expression. Multiply using (5b) and take the real part; you get

$$\tfrac{1}{5}e^x \cos 2x + \tfrac{2}{5}e^x \sin 2x.$$

In this differential equations course, we will make free use of complex exponentials in solving differential equations, and in doing formal calculations like the ones above. This is standard practice in science and engineering, and you need to get used to it.

Finding n-th roots.

To solve linear differential equations with constant coefficients, you need to be able find the real and complex roots of polynomial equations. Though a lot of this is done today with calculators and computers, one still has to know how to do an important special case by hand: finding the roots of

$$z^n = \alpha,$$

where α is a complex number, i.e., finding the n-th roots of α. Polar representation will be a big help in this.

Let's begin with a special case: the **n-th roots of unity**: the solutions to

$$z^n = 1\ .$$

To solve this equation, we use polar representation for both sides, setting $z = re^{i\theta}$ on the left, and using all possible polar angles on the right; using the exponential law to multiply, the above equation then becomes

$$r^n e^{in\theta} \;=\; 1 \cdot e^{(2k\pi i)}, \qquad k = 0, \pm 1, \pm 2, \dots\ .$$

Equating the absolute values and the polar angles of the two sides gives

$$r^n = 1, \qquad n\theta = 2k\pi\ , \qquad k = 0, \pm 1, \pm 2, \dots\ ,$$

from which we conclude that

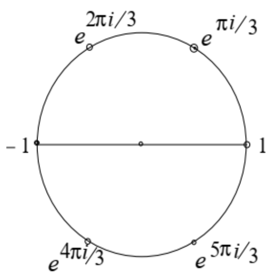
$$(*) \qquad r = 1, \qquad \theta \;=\; \frac{2k\pi}{n}\ , \qquad k = 0, 1, \dots, n-1\ .$$

In the above, we get only the value $r = 1$, since r must be real and non-negative. We don't need any integer values of k other than $0, \dots, n-1$ since they would not produce a complex number different from the above n numbers. That is, if we add an, an integer multiple of n, to k, we get the same complex number:

$$\theta' \;=\; \frac{2(k+an)\pi}{n} \;=\; \theta + 2a\pi; \qquad \text{and} \qquad e^{i\theta'} = e^{i\theta}, \quad \text{since } e^{2a\pi i} = (e^{2\pi i})^a = 1.$$

We conclude from $(*)$ therefore that

(21) *the n-th roots of 1 are the numbers* $e^{2k\pi i/n}, \quad k = 0, \dots, n-1.$



This shows there are n complex n-th roots of unity. They all lie on the unit circle in the complex plane, since they have absolute value 1; they are evenly spaced around the unit circle, starting with 1; the angle between two consecutive ones is $2\pi/n$. These facts are illustrated on the right for the case $n = 6$.

From (21), we get another notation for the roots of unity (ζ is the Greek letter "zeta"):

$$(22) \qquad \textit{the } n\textit{-th roots of 1 are} \quad 1, \zeta, \zeta^2, \dots, \zeta^{n-1}, \quad \text{where} \quad \zeta = e^{2\pi i/n}.$$

We now generalize the above to find the n-th roots of an arbitrary complex number w. We begin by writing w in polar form:

$$w \;=\; r\,e^{i\theta}\;; \qquad \theta = \operatorname{Arg} w,\; 0 \le \theta < 2\pi,$$

i.e., θ is the principal value of the polar angle of w. Then the same reasoning as we used above shows that if z is an n-th root of w, then

$$(23) \qquad z^n = w = r\,e^{i\theta}, \qquad \text{so} \qquad z = \sqrt[n]{r}\,e^{i(\theta+2k\pi)/n}, \qquad k = 0, 1, \dots, n-1.$$

Comparing this with (22), we see that these n roots can be written in the suggestive form

$$(24) \qquad \sqrt[n]{w} \;=\; z_0,\; z_0\zeta,\; z_0\zeta^2,\; \dots,\; z_0\zeta^{n-1}, \qquad \text{where} \;\; z_0 = \sqrt[n]{r}\,e^{i\theta/n}\;.$$

As a check, we see that all of the n complex numbers in (24) satisfy $z^n = w$:

$$\begin{aligned} (z_0\zeta^i)^n \;&=\; z_0^n\zeta^{ni} \;=\; z_0^n \cdot 1^i, && \text{since } \zeta^n = 1, \text{ by (22);} \\ &=\; w, && \text{by the definition (24) of } z_0 \text{ and (23).} \end{aligned}$$

Example 5. Find in Cartesian form all values of a) $\sqrt[3]{1}$ b) $\sqrt[4]{i}$.

Solution. a) According to (22), the cube roots of 1 are $1, \omega$, and ω^2, where

$$\begin{aligned} \omega \;&=\; e^{2\pi i/3} &&= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} &&= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ \omega^2 \;&=\; e^{-2\pi i/3} &&= \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} &&= -\frac{1}{2} - i\frac{\sqrt{3}}{2}\;. \end{aligned}$$

The greek letter ω ("omega") is traditionally used for this cube root. Note that for the polar angle of ω^2 we used $-2\pi/3$ rather than the equivalent angle $4\pi/3$, in order to take advantage of the identities

$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x\;.$$

Note that $\omega^2 = \bar{\omega}$. Another way to do this problem would be to draw the position of ω^2 and ω on the unit circle, and use geometry to figure out their coordinates.

b) To find $\sqrt[4]{i}$, we can use (24). We know that $\sqrt[4]{1} = 1, i, -1, -i$ (either by drawing the unit circle picture, or by using (22)). Therefore by (24), we get

$$\begin{aligned} \sqrt[4]{i} \;&= z_0,\; z_0 i,\; -z_0,\; -z_0 i, && \text{where } \; z_0 = e^{\pi i/8} \;=\; \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\;; \\ &= a+ib,\; -b+ia,\; -a-ib,\; b-ia, && \text{where } \; z_0 = a+ib \;=\; \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\;. \end{aligned}$$

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