

## Solving Ordinary Differential Equations – (ODE-IVPs) II

### Input-Response Models

**1. First-order linear ODE's with positive constant coefficient.** This is probably the most important first order equation; we use  $t$  and  $y$  as the variables, and think of the independent variable  $t$  as representing time. The IVP in standard form then is

$$(1) \quad y' + ky = q(t), \quad k > 0; \quad y(0) = y_0 .$$

The integrating factor for the ODE in (1) is  $e^{kt}$ ; using it, the general solution is

$$(2) \quad y = e^{-kt} \left( \int q(t)e^{kt} dt + c \right).$$

To get from this an explicit solution to (1), we change (cf. Notes D) the indefinite integral in (2) to a definite integral from 0 to  $t$ , which requires us to change the  $t$  in the integrand to a different dummy variable,  $u$  say; then the explicit solution to (1) is

$$(3) \quad y = e^{-kt} \int_0^t q(u)e^{ku} du + y_0 e^{-kt} .$$

In this form, note that the first term on the right is the solution to the IVP (1) corresponding to the initial condition  $y_0 = 0$ .

What we have done so far does not depend on whether  $k$  is positive or negative. However, the terminology we will now introduce makes sense only when  $k > 0$ , which we shall assume from now on.

Looking at (3) and assuming  $k > 0$ , we observe that as  $t \rightarrow \infty$ , the second term of the solution  $y_0 e^{-kt} \rightarrow 0$ , regardless of the initial value  $y_0$ . It is therefore called the **transient** since its effect on the solution dies away as time increases. As it dies away, what is left of the solution is the integral term on the right, which does not involve the initial value  $y_0$ ; it is called the **steady-state** or **long-term** solution to (1).

$$(4) \quad y = e^{-kt} \int_0^t q(u)e^{ku} du + y_0 e^{-kt}, \quad k > 0.$$

*steady-state*                      *transient*

Despite the use of the definite article, the steady-state solution is not unique: since all the solutions approach the steady-state solution as  $t \rightarrow \infty$ , they all approach each other, and thus any of them can be called the steady-state solution. In practice, it is usually the simplest-looking solution which is given this honor.

**2. Input-response; superposition of inputs.** When the ODE (1) is used to model a physical situation, the left-hand side usually is concerned with the physical set-up — the “system” — while the right-hand side represents something external which is driving or otherwise affecting the system from the outside. For this reason, the function  $q(t)$  is often called the **input**, or in some engineering subjects, the **signal**; the corresponding general solution (2) is called the **response** of the system to this input.

We will indicate the relation of input to response symbolically by

$$q(t) \rightsquigarrow y(t) \quad (\text{input} \rightsquigarrow \text{response}).$$

### Superposition principle for inputs.

For the ODE  $y' + ky = q(t)$ , let  $q_1(t)$  and  $q_2(t)$  be inputs, and  $c_1, c_2$  constants. Then

$$(5) \quad q_1 \rightsquigarrow y_1, \quad q_2 \rightsquigarrow y_2 \quad \implies \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2.$$

**Proof.** This follows in one line from the sum property of indefinite integrals; note that the proof does not require  $k$  to be positive.

$$y = e^{-kt} \int (q_1 + q_2)e^{kt} dt = e^{-kt} \int q_1 e^{kt} dt + e^{-kt} \int q_2 e^{kt} dt = y_1 + y_2 .$$

The superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces. Here is an easy example.

**Example 1.** Find the response of  $y' + 2y = q(t)$  to  $q = 1 + e^{-2t}$ .

**Solution.** The input  $q = 1$  generates the response  $y = 1/2$ , by inspection; the input  $e^{-2t}$  generates the response  $te^{-2t}$ , by solving; therefore the response to  $1 + e^{-2t}$  is  $1/2 + te^{-2t}$ .

### 3. Physical inputs; system responses to linear inputs.

We continue to assume  $k > 0$ . We want to get some feeling for how the system responds to a variety of inputs. The temperature model for (1) will be a good guide: in two notations – suggestive and neutral, respectively – the ODE is

$$(6) \quad T' + kT = kT_e(t), \quad y' + ky = kq_e(t) = q(t).$$

Note that the neutral notation writes the input in two different forms: the  $q(t)$  we have been using, and also in the form  $kq_e(t)$  with the  $k$  factored out. This last corresponds to the way the input appears in certain physical problems (temperature and diffusion problems, for instance) and leads to more natural formulas: for example,  $q_e$  and  $y$  have the same units, whereas  $q$  and  $y$  do not.

For this class of problems, the relation of response to input will be clearer if we relate  $y$  with  $q_e$ , rather than with  $q$ . We will use for  $q_e$  the generic name **physical input**, or if we have a specific model in mind, the *temperature* input, *concentration* input, and so on.

The expected behavior of the temperature model suggests general questions such as:

Is the response the same type of function as the physical input? What controls its magnitude?

Does the graph of the response lag behind that of the physical input?

What controls the size of the lag?

Our plan will be to get some feeling for the situation by answering these questions for several simple physical inputs. We begin with linear inputs. Throughout, keep the temperature model in mind to guide your intuition.

**Example 2.** Find the response of the system (6) to the physical inputs 1 and  $t$ .

**Solution.** The ODE is  $y' + ky = kq_e$ .

If  $q_e = 1$ , a solution by inspection is  $y = 1$ , so the response is 1.

If  $q_e = t$ , the ODE is  $y' + ky = kt$ ; using the integrating factor  $e^{kt}$  and subsequent integration by parts leads (cf. (2)) to the simplest steady-state solution

$$\begin{aligned} y &= e^{-kt} \int kte^{kt} dt \\ &= k e^{-kt} \left( \frac{te^{kt}}{k} - \frac{e^{kt}}{k^2} \right) \\ &= t - \frac{1}{k}. \end{aligned}$$

Thus the response of (6) is identical to the physical input  $t$ , but with a time lag  $1/k$ . This is reasonable when one thinks of the temperature model: the internal temperature increases linearly at the same rate as the temperature of the external water bath, but with a time lag dependent on the conductivity: the higher the conductivity, the shorter the time lag.

Using the superposition principle for inputs, it follows from Example 2 that for the ODE  $y' + ky = kq_e$ , its response to a general linear physical input is given by:

$$(7) \quad \text{linear input} \quad \text{physical input: } q_e = a + bt \quad \text{response: } a + b \left( t - \frac{1}{k} \right).$$

In the previous example, we paid no attention to initial values. If they are important, one cannot just give the steady-state solution as the response, one has to take account of them, either by using a definite integral as in (3), or by giving the value of the arbitrary constant in (2). Examples in the next section will illustrate.

#### 4. Response to discontinuous inputs, $k > 0$ .

The most basic discontinuous function is the unit-step function at a point, defined by

$$(8) \quad u_a(t) = \begin{cases} 0, & t < a; \\ 1, & t > a. \end{cases} \quad \text{unit-step function at } a$$

(We leave its value at  $a$  undefined, though some give it the value 0, others the value 1 there.)

**Example 3.** Find the response of the IVP  $y' + ky = kq_e$ ,  $y(0) = 0$ , for  $t \geq 0$ , to the unit-step physical input  $u_a(t)$ , where  $a \geq 0$ .

**Solution.** For  $t < a$  the input is 0, so the response is 0. For  $t \geq a$ , the steady-state solution for the physical input  $u_a(t)$  is the constant function 1, according to Example 2 or (7).

We still need to fit the value  $y(a) = 0$  to the response for  $t \geq a$ . Using (2) to do this, we get  $1 + ce^{-ka} = 0$ , so that  $c = -e^{ka}$ . We now assemble the results for  $t < a$  and  $t \geq a$  into one expression; for the latter we also put the exponent in a more suggestive form. We get finally

$$(9) \quad \text{unit-step input} \quad \text{physical input: } u_a(t), a \geq 0 \quad \text{response: } y(t) = \begin{cases} 0, & 0 \leq t < a; \\ 1 - e^{-k(t-a)}, & t \geq a. \end{cases}$$

Note that the response is just the translation  $a$  units to the right of the response to the unit-step input at 0.

Another way of getting the same answer would be to use the definite integral in (3); we leave this as an exercise.

As another example of discontinuous input, we focus on the temperature model, and obtain the response to the temperature input corresponding to the external bath initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by the ice-water bath.

## 5. Response to sinusoidal inputs.

Of great importance in the applications is the sinusoidal input, i.e., a pure oscillation like  $\cos \omega t$  or  $\sin \omega t$ , or more generally,  $A \cos(\omega t - \phi)$ . (The last form includes both of the previous two, as you can see by letting  $A = 1$  and  $\phi = 0$  or  $\pi/2$ ).

In the temperature model, this could represent the diurnal varying of outside temperature; in the concentration model, the diurnal varying of the level of some hormone in the bloodstream, or the varying concentration in a sewer line of some waste product produced periodically by a manufacturing process.

What follows assumes some familiarity with the vocabulary of pure oscillations: *amplitude, frequency, period, phase lag*.

### Response of $y' + ky = kq_e$ to the physical inputs $\cos \omega t$ , $\sin \omega t$ .

This calculation is a good example of how the use of complex exponentials can simplify integrations and lead to a more compact and above all more expressive answer. You should study it very carefully, since the ideas in it will frequently recur.

We begin by complexifying the inputs, the response, and the differential equation:

$$(15) \quad \cos \omega t = \operatorname{Re}(e^{i\omega t}), \quad \sin \omega t = \operatorname{Im}(e^{i\omega t});$$

$$(16) \quad \tilde{y}(t) = y_1(t) + iy_2(t);$$

$$(17) \quad \tilde{y}' + k\tilde{y} = ke^{i\omega t}.$$

If (16) is a solution to the complex ODE (17), then substituting it into the ODE and using the rule  $(u + iv)' = u' + iv'$  for differentiating complex functions gives

$$y_1' + iy_2' + k(y_1 + iy_2) = k(\cos \omega t + i \sin \omega t);$$

equating real and imaginary parts on the two sides gives the two real ODE's

$$(18) \quad y_1' + ky_1 = k \cos \omega t, \quad y_2' + ky_2 = k \sin \omega t;$$

this shows that the real and imaginary parts of our complex solution  $\tilde{y}(t)$  give us respectively the responses to the physical inputs  $\cos \omega t$  and  $\sin \omega t$ .

It seems wise at this point to illustrate the calculations with an example, before repeating them as a derivation. If you prefer, you can skip the example and proceed directly to the derivation, using the example as a solved exercise to test yourself afterwards.

**Example 5.** Find the response of  $y' + y = 0$  to the input  $\cos t$ ; in other words, find a solution to the equation  $y' + y = \cos t$ ; use complex numbers.

**Solution.** We follow the above plan and complexify the real ODE, getting

$$\tilde{y}' + \tilde{y} = e^{it} .$$

We made the right side  $e^{it}$ , since  $\cos t = \operatorname{Re}(e^{it})$ . We will find a complex solution  $\tilde{y}(t)$  for the complexified ODE; then  $\operatorname{Re}(\tilde{y})$  will be a real solution to  $y' + y = \cos t$ , according to (18) and what precedes it.

The complexified ODE is linear, with the integrating factor  $e^t$ ; multiplying both sides by this factor, and then following the steps for solving the first order linear ODE, we get

$$\tilde{y}' + \tilde{y} = e^{it} \Rightarrow (\tilde{y}e^t)' = e^{(1+i)t} \Rightarrow \tilde{y}e^t = \frac{1}{1+i} e^{(1+i)t} \Rightarrow \tilde{y} = \frac{1}{1+i} e^{it} .$$

This gives us our complex solution  $\tilde{y}$ ; the rest of the work is calculating  $\operatorname{Re}(\tilde{y})$ . To do this, we can use either the polar or the Cartesian representations.

Using the polar first, convert  $1+i$  to polar form and then use the reciprocal rule

$$1+i = \sqrt{2} e^{i\pi/4} \Rightarrow \frac{1}{1+i} = \frac{1}{\sqrt{2}} e^{-i\pi/4},$$

from which it follows from the multiplication rule (12a) that

$$\tilde{y} = \frac{1}{1+i} e^{it} = \frac{1}{\sqrt{2}} e^{i(t-\pi/4)}$$

and therefore our solution to  $y' + y = \cos t$  is

$$\operatorname{Re}(\tilde{y}) = \frac{1}{\sqrt{2}} \cos(t - \pi/4) ,$$

the pure oscillation with amplitude  $1/\sqrt{2}$ , circular frequency 1, and phase lag  $\pi/4$ .

Repeating the calculation, but using the Cartesian representation, we have

$$\tilde{y} = \frac{1}{1+i} e^{it} = \left( \frac{1-i}{2} \right) (\cos t + i \sin t)$$

whose real part is

$$\operatorname{Re}(\tilde{y}) = \frac{1}{2} (\cos t + \sin t) = \frac{\sqrt{2}}{2} \cos(t - \pi/4),$$

the last equality following from the sinusoidal identity

Instead of converting the Cartesian form to the polar, we could have converted the polar form to the Cartesian form by using the trigonometric addition formula. Since  $\cos \pi/4 = \sin \pi/4 = \sqrt{2}/2$ , it gives

$$\frac{1}{\sqrt{2}} \cos(t - \pi/4) = \frac{1}{\sqrt{2}} (\cos t \cos \pi/4 + \sin t \sin \pi/4) = \frac{1}{2} (\cos t + \sin t)$$

However, in applications the polar form of the answer is generally preferred, since it gives directly important characteristics of the solution — its amplitude and phase lag, whereas these are not immediately apparent in the Cartesian form of the answer.

Resuming our general solving of the ODE (17) using the circular frequency  $\omega$  and the constant  $k$ , but following the same method as in the example, the integrating factor is  $e^{kt}$ ; multiplying through by it leads to

$$(\tilde{y}e^{kt})' = ke^{(k+i\omega)t},$$

integrate both sides, multiply through by  $e^{-kt}$ , and scale the coefficient to lump constants and make it look better:

$$(19) \quad \tilde{y} = \frac{k}{k+i\omega} e^{i\omega t} = \frac{1}{1+i(\omega/k)} e^{i\omega t}.$$

This is our complex solution.

Writing  $1 + i(\omega/k)$  in polar form, then using the reciprocal rule we have

$$(20) \quad 1 + i(\omega/k) = \sqrt{1 + (\omega/k)^2} e^{i\phi} \Rightarrow \frac{1}{1 + i(\omega/k)} = \frac{1}{\sqrt{1 + (\omega/k)^2}} e^{-i\phi},$$

where  $\phi = \tan^{-1}(\omega/k)$ .

Therefore in polar form,

$$(21) \quad \tilde{y} = \frac{1}{\sqrt{1 + (\omega/k)^2}} e^{i(\omega t - \phi)}.$$

The real and imaginary parts of this complex response give the responses of the system to respectively the real and imaginary parts of the complex input  $e^{i\omega t}$ ; thus we can summarize our work as follows:

**First-order Sinusoidal Input Theorem.** For the equation  $y' + ky = kq_e$  we have

$$(22) \quad \text{physical input: } \begin{cases} q_e = \cos \omega t \\ q_e = \sin \omega t \end{cases} \quad \text{response: } \begin{cases} y_1 = A \cos(\omega t - \phi) \\ y_2 = A \sin(\omega t - \phi) \end{cases},$$

$$(23) \quad A = \frac{1}{\sqrt{1 + (\omega/k)^2}}, \quad \phi = \tan^{-1}(\omega/k).$$

The calculation can also be done in Cartesian form as in Example 5, then converted to polar form using the sinusoidal identity. We leave this as an exercise.

The Sinusoidal Input Theorem is more general than it looks; it actually covers any sinusoidal input  $f(t)$  having  $\omega$  as its circular frequency. This is because any such input can be written as a linear combination of  $\cos \omega t$  and  $\sin \omega t$ :

$$f(t) = a \cos \omega t + b \sin \omega t, \quad a, b \text{ constants}$$

and then it is an easy exercise to show:

$$(22') \quad \text{physical input: } q_e = \cos \omega t + b \sin \omega t \quad \text{response: } a y_1 + b y_2 .$$

## References and cited materials

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