Logic in Computer Science Midterm

1. **Induction and Recursion**

   Consider augmenting the alphabet of propositional logic with two new symbols: $\top$ and $\bot$, where $\top$ is a symbol that can be used wherever a propositional symbol can be used but always evaluates to true and $\bot$ is a similar symbol except that it always evaluates to false.

   (a) Give an inductive definition of propositional formulas including $\top$ and $\bot$. You may restrict the usual set of five Boolean connectives to any complete set. The formulas should be freely generated (but you don't have to prove that they are).

   (b) Let W_{tf} be the set of propositional formulas defined inductively in part (a). Define a recursive function *simp* which maps formulas in W_{tf} to a logically equivalent formula in $W \cup \{\top, \bot\}$, where W is the subset of formulas in W_{tf} that do not contain $\top$ and $\bot$ (i.e. *simp* either returns a formula that does not contain $\top$ or $\bot$ or it returns $\top$ or $\bot$).

2. **Decidability and Semi-Decidability**

   (a) Prove that the union of two effectively enumerable sets is again effectively enumerable.

   (b) Prove that the intersection of two effectively enumerable sets is again effectively enumerable.

3. **First-Order Logic: Proofs and Models**

   Following are several instances of $\Gamma \vdash \phi$. For each instance, either prove that $\Gamma \vdash \phi$ (either give an actual formal deduction or prove that one exists) or give a model and variable assignment which satisfies Γ but not ϕ (showing that $\Gamma \not\models \phi$ and therefore, by soundness, $\Gamma \nvdash \phi$).

   (a) $\emptyset \vdash \neg\forall\, y\, \exists\, x\, (Py \wedge \neg Px)$

   (b) $\forall\, x\, (x = a) \vdash \forall\, y\, \forall\, z\, (y = z)$

   (c) $\forall\, x\, (Px \rightarrow \neg Qx) \vdash (\exists\, y\, Qy \rightarrow \exists\, z\, \neg Pz)$

   (d) $\forall\, x\, \forall\, y\, \forall\, z\, (Pxy \rightarrow Pyz \rightarrow Pxz) \vdash \forall\, x\, \forall\, y\, (Pxy \rightarrow Pyx)$

4. **Homomorphisms and Definability**

(a) Show that if A is a substructure of B and $\models_A \phi[s]$ for some quantifier-free formula ϕ and variable assignment s, then $\models_B \phi[s]$.

(b) A formula is *existential* if it is of the form $\exists x_1 \cdots \exists x_n\, \theta$, where θ is quantifier-free. Show that if A is a substructure of B and $\models_A \phi[s]$ for some existential formula ϕ and variable assignment s, then $\models_B \phi[s]$.

(c) Let Σ be a signature with equality and a single binary predicate symbol: $<$. Let M be a Σ-model whose domain is the set of positive integers and which interprets $<$ as follows: $x <^M y$ iff x divides y. Show that the set of prime numbers is definable in M.

5. **First-Order Compactness**

(a) Suppose that Γ is a set of sentences and τ is a sentence. Show that if $Mod\,\Gamma = Mod\,\tau$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $Mod\,\Gamma_0 = Mod\,\tau$.

(b) Recall that a class $\mathcal{K}$ of models is first-order definable iff $\mathcal{K} = Mod\,\tau$ for some sentence τ. Show that the class of all infinite groups is not first-order definable. You may use the fact that there are arbitrarily large finite groups.