**INSTRUCTION:** THIS IS A TAKE AWAY C.A.T.

ATTEMPT ALL THE QUESTIONS.

GROUP DISCUSSIONS ALLOWED.

**QUESTION 1.**

Consider a system of two light rods of equal length, smoothly jointed together, with the other two ends of the rods fixed at two points in a horizontal plane. A mass $m$ is attached to the point where the rods are jointed. Introduce an appropriate generalized coordinate for the system, and determine the Lagrangian.

**QUESTION 2.**

A *double pendulum* consists of a simple pendulum of mass $m_1$ and length $l_1$ pivoted at the origin, together with another simple pendulum of mass $m_2$ and length $l_2$, pivoted at the mass $m_1$. The whole system moves freely in a vertical plane under gravity. If $\theta_1$ and $\theta_2$ denote the angles each pendulum makes with the vertical, show that the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\ &\quad + m_1 g l_1\cos\theta_1 + m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1)\ . \end{aligned}$$

**QUESTION 3.**

In lectures we considered the example of a bead of mass $m$ sliding freely on a circular wire of radius $a$, with the wire forced to rotate about a vertical diameter with constant angular frequency $\dot{\varphi} = \omega$. The effective Lagrangian for the angle $\theta = \theta(t)$ is

$$L_1(\theta, \dot{\theta}) = \frac{1}{2}ma^2\dot{\theta}^2 - V_{\text{eff}}(\theta)\ ,$$

where

$$V_{\text{eff}}(\theta) = mga\cos\theta - \frac{1}{2}ma^2\omega^2\sin^2\theta\ .$$

Show that $\theta = 0$, $\theta = \pi$ and (for $\omega \geq \sqrt{g/a}$) $\theta = \theta_0$ given by

$$\theta_0 = \cos^{-1}\left(-\frac{g}{\omega^2 a}\right)$$

are equilibria, and determine which are stable and which are unstable.

**QUESTION 4.**

Consider $N$ point particles with masses $m_I$, position vectors $\mathbf{r}_I$ and Lagrangian

$$L = \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 - V(\{|\mathbf{r}_I - \mathbf{r}_J|\}) \ .$$

In particular the potential function $V$ depends only on the distances between the particles. Show that the Galilean boost $\mathbf{r}_I \to \mathbf{r}_I + \epsilon \mathbf{v} t$ (for all $I = 1, \ldots, N$) is a symmetry of $L$, and hence identify the function $f$ that enters the form of Noether's theorem given in lectures. Does this lead to a new conserved quantity?

**QUESTION 5.**

A bead of mass $m_1$ slides without friction on a fixed horizontal wire which occupies the interval $[-a, a]$ of the $x$-axis. A light spring, of spring constant $k$, connects the bead to the point $-a$, and a second light spring, with the same spring constant, connects the bead to the point $a$. A massless rod of length $l$ hangs freely from the bead and its other end carries a particle of mass $m_2$. The motion is restricted to the vertical plane containing the wire.

(a) Show that the Lagrangian for the system is

$$L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + 2l \cos\theta \, \dot{x}\dot{\theta} + l^2 \dot{\theta}^2) - kx^2 + m_2 g l \cos\theta \ ,$$

where $x$ is the position of the bead on the wire, $\theta$ is the angle between the rod and the downward vertical, and $g$ is the acceleration due to gravity.

(b) Find a constant of the motion.

(c) Determine the Lagrange equations of motion and show there are two equilibrium points.

(d) Write down the quadratic Lagrangian for small oscillations about the point of stable equilibrium.

(e) Find the normal frequencies when $g = l = k = m_1 = m_2 = 1$.

**QUESTION 6.**

A thin uniform disc of radius $a$ and mass $M$ moves on a smooth horizontal table, touching the table at a point of its circumference. Introduce Cartesian coordinates $(x, y, z)$ for the centre of mass of the disc, and Euler angles $(\theta, \varphi, \psi)$ to describe its orientation, with the $\mathbf{e}_3$ axis parallel to the axis of symmetry of the disc, so that $\theta$ gives the angle between the plane of the disc and the table.

(a) Explain why there are five degrees of freedom for this system, and why you can choose $(x, y, \theta, \varphi, \psi)$ as generalized coordinates.

(b) Write down the Lagrangian. Show that $\dot{x}$ and $\dot{y}$ are both constant.

(c) Initially the disc has spin $n$ about its axis of symmetry, which makes an angle $\alpha$ with the vertical, while $\dot{\theta} = 0 = \dot{\varphi}$. Show that, in the subsequent motion, the spin around the axis is constant and

$$a\dot{\theta}^2(1 + 4\cos^2\theta) + 4an^2(\cos\alpha - \cos\theta)^2(\sin\theta)^{-2} + 8g(\sin\theta - \sin\alpha) \;=\; 0\ .$$

**QUESTION 7.**

Determine the Hamiltonian for the Lagrange top.

**QUESTION 8.**

Consider a rotating rigid body with centre of mass coordinates $(x, y, z)$, principal moments of inertia $I_1, I_2, I_3$ about the centre of mass, and mass $M$. Explain why, in terms of Euler angles $(\theta, \varphi, \psi)$, the kinetic energy of the body is

$$\begin{aligned} T \;=\; & \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}I_1(\dot{\theta}\,\sin\psi - \dot{\varphi}\,\sin\theta\cos\psi)^2 \\ & + \frac{1}{2}I_2(\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\,\cos\theta)^2\ . \end{aligned}$$

**QUESTION 9.**

Consider the system of Euler equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \ , \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \ , \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 \ , \end{aligned}$$

describing the free rotation of a rigid body with principal moments of inertia $I_1, I_2, I_3$. Suppose that $I_1 < I_2$, $I_3 = I_1 + I_2$ and the body is set in motion with $\omega_2 = 0$ and $\omega_3\sqrt{I_2 + I_1} = \omega_1\sqrt{I_2 - I_1}$. Show that

$$\dot{\omega}_1^2 = \left(\frac{I_2 - I_1}{I_2 + I_1}\right)\omega_1^2\left(\frac{2T}{I_2} - \omega_1^2\right) \ ,$$

where $T$ is the conserved kinetic energy. Hence find a solution of the form $\omega_1(t) = c_1 \operatorname{sech}(c_2 t)$ for appropriate constants $c_1$, $c_2$. What happens as $t \to \infty$?

**QUESTION 10.**

(a) Show that none of the three principal moments of inertia can exceed the sum of the other two.

(b) By introducing cylindrical polar coordinates, show that the centre of mass of an axi-symmetric body lies on the axis of symmetry. Show that the axis of symmetry is a principal axis, and that if we take this to be the $\mathbf{e}_3$ direction in an orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ then the other two principal moments of inertia satisfy $I_1 = I_2$.

(c) Consider a rigid straight rod of line density $\rho$. Taking the rod to lie along the $\mathbf{e}_3$ direction, show that the inertia tensor at any point on the rod in this basis is diagonal, with eigenvalues $I_1 = I_2$, $I_3 = 0$. More generally when is it possible to have zero as a principal moment of inertia?

**QUESTION 11.**

Find the principal moments of inertia at the centre of mass of an ellipsoid of mass $M$ and uniform density, bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 ,$$

where $(x, y, z)$ are standard Cartesian coordinates. Using your result hence deduce the inertia tensor of a uniform sphere at a point on its surface.

**QUESTION 12.**

Consider the *elliptic cylindrical coordinates* on $\mathbb{R}^3$, related to Cartesian coordinates $(x, y, z)$ by

$$x = a \cosh \mu \cos \nu , \qquad y = a \sinh \mu \sin \nu , \qquad z = z .$$

Show that the Hamiltonian for a particle of mass $m$ moving in a potential $V$ is given in these coordinates by

$$H = \frac{1}{2ma^2(\sinh^2 \mu + \sin^2 \nu)}(p_\mu^2 + p_\nu^2) + \frac{1}{2m}p_z^2 + V(\mu, \nu, z) .$$

Assuming the potential $V$ takes the form

$$V(\mu, \nu, z) = \frac{V_\mu(\mu) + V_\nu(\nu)}{\sinh^2 \mu + \sin^2 \nu} + V_z(z) ,$$

show that the Hamilton-Jacobi equation is completely separable, and determine the three corresponding ordinary differential equations.

**QUESTION 13.**

(*Optional: this question is included for interest.*) Consider the central inverse square law force Lagrangian

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{\kappa}{r} ,$$

where $r = |\mathbf{r}|$ and $\kappa$ is a constant.

(a) Show by direct computation that the vector

$$\mathbf{A} = \mathbf{p} \wedge \mathbf{L} - m\kappa\frac{\mathbf{r}}{r}$$

is conserved, where $\mathbf{p} = m\dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ are momentum and angular momentum about the origin, respectively. [*Hint*: You may find it helpful to write $\dot{\mathbf{p}} = h(r)\frac{\mathbf{r}}{r}$ and show $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \wedge \mathbf{L}) = -mh(r)r^2\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathbf{r}}{r})$.]

(b) Show that $\mathbf{A} \cdot \mathbf{L} = 0$ and $|\mathbf{A}|^2 = 2mE|\mathbf{L}|^2 + m^2\kappa^2$, where $E$ is the conserved energy. Explain why $\mathbf{A}$ is a constant vector in the plane of the orbit.

(c) By taking the dot product $\mathbf{A} \cdot \mathbf{r}$ derive the orbit equation

$$\frac{1}{r} = \frac{m\kappa}{|\mathbf{L}|^2}\left(1 + \frac{|\mathbf{A}|}{m\kappa}\cos\theta\right) ,$$

where $\theta$ denotes the angle between $\mathbf{r}$ and $\mathbf{A}$. Notice we have found the orbit without solving any differential equation! The eccentricity is $|\mathbf{A}|/m\kappa$.