INSTRUCTION: ATTEMPT ALL THE QUESTIONS.

THIS WILL ACCOUNT FOR 20% OF THE FINAL GRADES

MAKE SURE YOU HAVE A WORKING SCIENTIFIC CALCULATOR

TIME: ONE AND A HALF HOURS

QUESTION 1.

(a) Show that the Poisson bracket relations

$$\{\mathbf{q}, f\} = \frac{\partial f}{\partial \mathbf{p}} \quad \text{and} \quad \{\mathbf{p}, f\} = -\frac{\partial f}{\partial \mathbf{q}}$$

hold for any function f on phase space.

(b) Prove the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 .$$

QUESTION 2.

(a) Show that the angular momentum vector $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ satisfies the Poisson bracket relation $\{L_i, L_j\} = \sum_{k=1}^{3} \epsilon_{ijk} L_k$ [*Hint*: you may use the identity $\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$]. Deduce that $\{\mathbf{L}, |\mathbf{L}|^2\} = \mathbf{0}$.

(b) Show that for a particle moving in a central potential $V = V(|\mathbf{r}|)$ we have $\{\mathbf{L}, H\} = 0$. Deduce that angular momentum is conserved.

(c) Recall that the Laplace-Runge-Lenz vector is defined as

$$\mathbf{A} \equiv \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} ,$$

where κ is a constant. Derive the Poisson bracket relations $\{L_i, A_j\} = \sum_{k=1}^{3} \epsilon_{ijk} A_k$.

(d) (*Optional: for the enthusiast.*) Assuming that the Hamiltonian is $H = |\mathbf{p}|^2/2m - \kappa/|\mathbf{r}|$, show that $\{\mathbf{A}, H\} = \mathbf{0}$ and deduce that $\mathbf{A}$ is conserved.

QUESTION 3.

(a) Show that $Q = \arctan\frac{q}{p}$, $P = \frac{1}{2}(p^2 + q^2)$ is a canonical transformation.

(b) For which functions $f(q)$ does $Q = f(q)\,\mathrm{e}^t\cos p$, $P = f(q)\,\mathrm{e}^{-t}\sin p$ define a canonical transformation?

QUESTION 4.

Let

$$\mathcal{D}_f \;\equiv\; \sum_{\alpha=1}^{2n} \frac{\partial f}{\partial y_\alpha}\Omega_{\alpha\beta}\frac{\partial}{\partial y_\beta}$$

be the Hamiltonian vector field associated to the function $f = f(\mathbf{y})$ on phase space, with coordinates $\mathbf{y} = (y_1, \ldots, y_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and where Ω is the symplectic matrix introduced in lectures.

(a) Show that for all functions f, g, h on phase space we have

$$\mathcal{D}_f(\mathcal{D}_g\, h) - \mathcal{D}_g(\mathcal{D}_f\, h) \;=\; \mathcal{D}_{\{f,g\}}\, h\ .$$

(b) Consider the one-dimensional harmonic oscillator Hamiltonian $H(q,p) = \frac{1}{2}(p^2 + q^2)$. Show that

$$\mathrm{e}^{-t\mathcal{D}_H}\, q \;=\; q\cos t + p\sin t\ , \qquad \mathrm{e}^{-t\mathcal{D}_H}\, p \;=\; p\cos t - q\sin t\ .$$

Deduce that $Q = q\cos s + p\sin s$, $P = p\cos s - q\sin s$ defines a canonical transformation for any s. What happens for $s = \pi/2$ and $s = \pi$?