FINAL EXAMINATION

INSTRUCTION: ATTEMPT ALL THE QUESTIONS IN **SECTION A** AND ANY OTHER FOUR IN **SECTION B.**

THIS WILL ACCOUNT FOR 70% OF THE FINAL GRADES. MAKE SURE YOU HAVE A WORKING SCIENTIFIC CALCULATOR. OBSERVE TOTAL SILENCE IN THE EXAMINATION ROOM.

TIME: THREE HOURS

SECTION A

QUESTION 1. (20 MARKS)

A smooth rod OP is forced to rotate with angular speed ω about a fixed vertical axis OQ so that it makes a constant angle α with OQ, where $0 < \alpha < \frac{\pi}{2}$. A ring of mass m slides freely on the rod OP.

(a) Using a coordinate r for the distance of the ring to O, find the Lagrangian for the dynamics of the ring. Deduce that, as long as the ring remains on the rod, the quantity

$$H = \frac{1}{2}m(\dot{r}^2 - \omega^2 r^2 \sin^2\alpha) + mgr\cos\alpha$$

is a constant of the motion.

(b) The ring is projected from O towards P with initial speed $\frac{\lambda g}{\omega}\cot\alpha$. Show that if $0 < \lambda < 1$ the ring can never exceed a maximum value of r. Setting $\lambda = 1$ find $r(t)$.

SECTION B

QUESTION 2. (20 MARKS)

A light string is stretched to a tension τ between two fixed points A and B, a distance $3a$ apart, on a smooth horizontal table. Two point masses, each of mass m, are attached to the string at the points P_1, P_2. In equilibrium the four points A, P_1, P_2 and B are all equal distances apart. The system is set to perform small transversal oscillations by displacing transversely the two masses. Given that the potential energy is $\tau\Delta$, where Δ is the extension of the string from equilibrium and τ is constant, find the normal frequencies and normal modes of the vibration. Sketch the normal modes. [*Hint*: You might find it helpful to introduce generalized coordinates x, y such that $A = (0, 0)$, $P_1 = (a, x)$, $P_2 = (2a, y)$, $B = (3a, 0)$.]

QUESTION 3. (20 MARKS)

(a) Let $\mathcal{S}$, $\hat{\mathcal{S}}$ and $\mathcal{S}'$ be three reference frames, all with the same origin O. By introducing appropriate rotation matrices show that if the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is $\boldsymbol{\omega}$, and in turn $\hat{\mathcal{S}}$ has angular velocity $\hat{\boldsymbol{\omega}}$ relative to $\mathcal{S}'$, then $\mathcal{S}$ has angular velocity $\boldsymbol{\omega} + \hat{\boldsymbol{\omega}}$ relative to $\mathcal{S}'$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a time-dependent orthonormal basis. Explain why $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_1 = 0$, and give two similar relations. Hence deduce that for some $\alpha, \beta, \gamma, \lambda, \mu, \nu$,

$$\dot{\mathbf{e}}_1 = \beta\mathbf{e}_3 - \gamma\mathbf{e}_2 , \qquad \dot{\mathbf{e}}_2 = \lambda\mathbf{e}_1 - \alpha\mathbf{e}_3 , \qquad \dot{\mathbf{e}}_3 = \mu\mathbf{e}_2 - \nu\mathbf{e}_1 .$$

Next show that $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + \dot{\mathbf{e}}_2 \cdot \mathbf{e}_1 = 0$, and deduce from this and similar identities that $\lambda = \gamma$, $\mu = \alpha$, and $\beta = \nu$. Hence deduce that there exists a vector $\boldsymbol{\omega}$ such that $\dot{\mathbf{e}}_i = \boldsymbol{\omega} \wedge \mathbf{e}_i$ holds for all $i = 1, 2, 3$. [*This is an alternative way to introduce the angular velocity vector* $\boldsymbol{\omega}$.]

QUESTION 4. (20 MARKS)

Consider a closed system consisting of N point particles with masses m_I, position vectors $\mathbf{r}_I$ in an inertial frame $\mathcal{S}$, such that particle J exerts a force $\mathbf{F}_{IJ}$ on particle I for $I \neq J$.

(a) Explain why Newton's third law and Galilean invariance means there exists an inertial frame $\mathcal{S}_0$ where the centre of mass

$$\mathbf{R} \;=\; \frac{\sum_{I=1}^{N} m_I \mathbf{r}_I}{\sum_{I=1}^{N} m_I}$$

is at rest at the origin.

(b) Suppose now that there exist functions $V_{IJ} = V_{JI} = V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$, depending only on the distances $|\mathbf{r}_I - \mathbf{r}_J|$ between pairs of particles, such that $\mathbf{F}_{IJ} = -\partial_{\mathbf{r}_I} V_{IJ}$. Show that the total angular momentum and total energy

$$E \;=\; \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 + \sum_{I<J} V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$$

are conserved.

QUESTION 5. (20 MARKS)

Consider the one-dimensional harmonic oscillator with action $S[q(t)] = \int_0^{2\pi} L(q, \dot{q})\, \mathrm{d}t$, Lagrangian

$$L \;=\; L(q, \dot{q}) \;=\; \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m q^2 \;,$$

and with boundary conditions $q(0) = q(2\pi) = 1$. Determine the Lagrange equation of motion, and show that a solution is $q(t) = \cos t$. Is this solution unique? By considering the two paths $q_1(t) = 1$, $q_2(t) = \cos 2t$ show that the critical function $q(t)$ is neither a maximum nor a minimum of the action S.

QUESTION 6. (20 MARKS)

Consider the change of generalized coordinates $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$. Show that

$$\dot{q}_a = \sum_{b=1}^{n} \frac{\partial q_a}{\partial \tilde{q}_b} \dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t}.$$

Defining

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \equiv L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) ,$$

show that the Lagrange equations in the two coordinate systems are related via

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} = \sum_{b=1}^{n} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right] \frac{\partial q_b}{\partial \tilde{q}_a} .$$

Hence conclude that the Lagrange equations take the same form in all coordinate systems.

QUESTION 7. (20 MARKS)

Consider the purely kinetic Lagrangian

$$L = T = \frac{1}{2} \sum_{a,b=1}^{n} g_{ab}(\mathbf{q}) \dot{q}_a \dot{q}_b ,$$

where we assume that the symmetric matrix $g_{ab} = g_{ba}$ depends on the generalized coordinates $\mathbf{q}$ and is invertible at each point in configuration space. Show that Lagrange's equations take the form

$$\ddot{q}_a + \sum_{b,c=1}^{n} \Gamma^a_{bc} \dot{q}_b \dot{q}_c = 0 , \qquad a = 1, \ldots, n ,$$

where

$$\Gamma^a_{bc} \equiv \frac{1}{2} \sum_{d=1}^{n} (g^{-1})^{ad} \left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d}\right) .$$

The matrix of functions $g_{ab}(\mathbf{q})$ defines a *metric* on configuration space, and the Lagrange equation is known as the *geodesic equation.*