Classical Mechanics

Appendix: Levi-Civita alternating symbol

The *Levi-Civita alternating symbol* ϵ_{ijk} is defined by

$$\epsilon_{ijk} \equiv \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123\ , \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Notice that $\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$ (cyclically permuting the indices), while $\epsilon_{ijk} = -\epsilon_{ikj}$. With this definition the cross product $\mathbf{A} \wedge \mathbf{B}$ of two vectors $\mathbf{A}$, $\mathbf{B}$ has i^{th} component

$$(\mathbf{A} \wedge \mathbf{B})_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k\ .$$

A useful identity is

$$\epsilon_{ijk}\epsilon_{abc} = \det \begin{pmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} \\ \delta_{ja} & \delta_{jb} & \delta_{jc} \\ \delta_{ka} & \delta_{kb} & \delta_{kc} \end{pmatrix}\ ,$$

where the *Kronecker delta symbol* is defined by

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\ .$$

As a special case we have the identity

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ibc} = \delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb}\ .$$

As an application, let us compute the scalar quadruple product:

$$\begin{aligned} (\mathbf{A} \wedge \mathbf{B}) \cdot (\mathbf{C} \wedge \mathbf{D}) &= \sum_{i,j,k,b,c=1}^{3} \epsilon_{ijk} A_j B_k\, \epsilon_{ibc} C_b D_c \\ &= \sum_{i,j,k,b,c=1}^{3} (\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb})\, A_j B_k C_b D_c \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})\ . \end{aligned}$$

Other vector product identities are similarly straightforward to derive (for example the scalar triple product identity follows immediately from cyclically permuting the indices on ϵ_{ijk}). Although we won't use the following identity, we present it for interest:

$$(\det M)\, \epsilon_{ijk} = \sum_{a,b,c=1}^{3} \epsilon_{abc} M_{ia} M_{jb} M_{kc}\ ,$$

where $M = (M_{ij})$ is any 3×3 matrix.