

# LOGIC IN COMPUTER SCIENCE

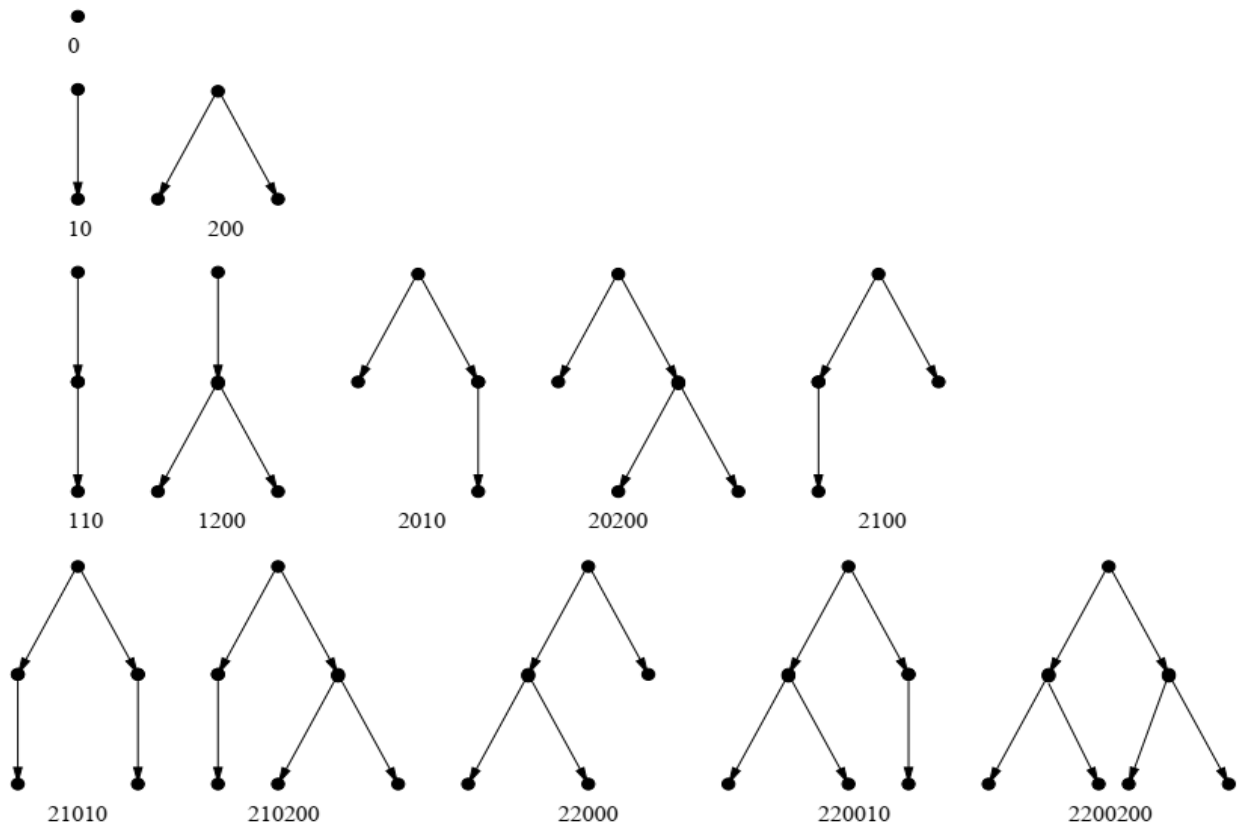
## FINAL EXAM

### ATTEMPT ANY FIVE QUESTIONS

**TIME: 3 Hours**

#### QUESTION ONE: [20 Marks]

Consider the set of directed, rooted, ordered binary trees in which any node may have 0, 1, or 2 children. These trees can be represented by expressions over the alphabet  $\{0, 1, 2\}$  as follows. The tree with a single vertex is represented by the expression 0. Any other tree is represented by starting with the digit corresponding to the number of children at the root node and then concatenating the expressions corresponding to the sub-trees for each child in order. All such trees of height 2 or less are shown below with their corresponding expressions.



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- (a) Give an inductive definition of the set  $T$  of expressions which represent such trees.
- (b) Prove that  $T$  is freely generated. You will need to prove a lemma similar to the one proved in class (and in the solutions to last year's problems).
- (c) Prove that 1020 and 2210 are not in  $T$ .
- (d) The *height* of such a tree is defined to be the longest possible path starting at the root node. For example, the tree 0 has height 0. The trees 10 and 200 have height 1, and the rest of the trees shown above have height 2. Give a recursive definition of a function which returns the height of any tree in  $T$ .

## QUESTION TWO: [20 Marks]

- 1. Prove that for all wff's  $\alpha$ ,  $\alpha$  is satisfiable iff  $\neg\alpha$  is not valid.
- 2. Is the following formula valid? Justify your answer using a truth table:

$$(((P \wedge Q) \rightarrow R) \rightarrow S) \rightarrow (((R \rightarrow Q) \rightarrow P) \rightarrow S).$$

- 3. Show that  $\{\downarrow\}$  is complete.
- 4. Say that a set  $\Sigma_1$  of wffs is *equivalent* to a set  $\Sigma_2$  of wffs iff for any wff  $\alpha$ , we have  $\Sigma_1 \models \alpha$  iff  $\Sigma_2 \models \alpha$ . A set  $\Sigma$  is *independent* iff no member of  $\Sigma$  is tautologically implied by the remaining members in  $\Sigma$ . Prove that every finite set of wffs has an independent equivalent subset.

## QUESTION THREE: [20 Marks]

- 1. Prove that if  $\phi$  and  $\psi$  are *first-order wffs*, then  $\phi \models \psi$  iff  $\models (\phi \leftrightarrow \psi)$ .
- 2. Suppose  $P$  is a binary predicate. Show that no one of the following sentences is logically implied by the other two. Do this by giving a model for each sentence in which the sentence is false but the other two sentences are true.

- (a)  $\forall x Pxx$
- (b)  $\forall x \forall y (Pxy \vee Pyx \vee x = y)$
- (c)  $\exists x \forall y Pxy$

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## QUESTION FOUR: [20 Marks]

1.

Consider a language with equality and a single binary predicate symbol  $P$ . For each set  $\mathcal{M}$  of models below, write a first order sentence  $\phi$  such that  $\models_M \phi$  iff  $M \in \mathcal{M}$ .

- (a)  $\mathcal{M} = \{M \mid P^M \text{ is a transitive relation} \}$ .
- (b)  $\mathcal{M} = \{M \mid P^M \text{ defines a function} \}$ .
- (c)  $\mathcal{M} = \{M \mid P^M \text{ is a bijection (i.e. a function that is 1-1 and onto)} \}$ .

2.

Consider a signature  $\Sigma$  with no constant symbols, no predicate symbols (except for equality), and a single binary function symbol,  $+$ . Let  $M$  be a  $\Sigma$ -model with domain  $\mathbf{N}$  (the natural numbers) which interprets  $+$  in the standard way.

Note that the only non-logical symbols you may use are  $=$  and  $+$ .

- (a) Give a  $\Sigma$ -formula which defines the set  $\{0\}$  in  $M$ .
- (b) Give a  $\Sigma$ -formula which defines the set  $\{1\}$  in  $M$ .
- (c) Give a  $\Sigma$ -formula which defines the binary relation  $\{\langle m, n \rangle \mid m < n\}$  in  $M$ .

## QUESTION FIVE: [20 Marks]

- 1. Give a deduction (from  $\emptyset$ ) of  $\forall x Pxx \rightarrow y = z \rightarrow Pyz$ .
- 2. Assume that  $x$  does not occur free in  $\alpha$ . Show that

$$\vdash (\alpha \rightarrow \exists x \beta) \leftrightarrow \exists x (\alpha \rightarrow \beta)$$

(i.e. show  $\vdash (\alpha \rightarrow \exists x \beta) \rightarrow \exists x (\alpha \rightarrow \beta)$  and  $\vdash \exists x (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \exists x \beta)$ ). Note that you do not have to give the deduction, just prove that one exists.

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3. A complete calculus has the property that each sentence either has a deduction (from  $\emptyset$ ) or a counter-model (i.e., a model in which it is false). For each of the following sentences, either show there is a deduction or give a counter-model.

(a)  $\forall x (Qx \rightarrow \forall y Qy)$

(b)  $\forall z (Pz \rightarrow Qz) \rightarrow (\exists x Px \rightarrow \forall y Qy)$

(c)  $\neg \exists y \forall x (Pxy \leftrightarrow \neg Pxx)$

Assume that  $A \leftrightarrow B$  is an abbreviation for  $\neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$ .

## QUESTION SIX: [20 Marks]

1. Show that the following sentence is finitely valid (i.e. true in every finite model). Hint: show that any model of the negation must be infinite.

$$\exists x \forall y \exists z [(Qzx \rightarrow Qzy) \rightarrow (Qxy \rightarrow Qxx)]$$

2. Let  $\phi$  be the sentence  $\forall x \forall y \forall z (Pxy \rightarrow Pyz \rightarrow Pxz)$ . Recall that a theory is (the deductive closure of) a set of sentences. Give an example of a theory  $T$  such that
- (a)  $\phi$  is  $T$ -valid,
  - (b)  $\phi$  is  $T$ -satisfiable but not  $T$ -valid,
  - (c)  $\phi$  is  $T$ -unsatisfiable

## QUESTION SEVEN: [20 Marks]

1.

Assume that Augustine is loved by everyone who loves someone. Also assume that no one loves nobody. We wish to deduce that Augustine is loved by everyone.

- (a) Consider a language with one constant symbol  $a$  (for Augustine) and a 2-place predicate symbol  $L$ , so that  $L(x, y)$  encodes “x loves y”. Write three first-order sentences which encode the two premises and the conclusion.
- (b) To show that the premises imply the conclusion, we will show that it is impossible to satisfy the premises and the negation of the conclusion. Skolemize each premise and the negation of the conclusion to obtain three clauses.
- (c) Use first-order resolution to show that the empty clause is obtainable from the original three clauses.

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## 2.

Recall that  $b$  is a *sequence number* iff for some  $m \geq -1$  and some  $a_0, \dots, a_m$ ,  $b = \langle a_0, \dots, a_m \rangle$ , where  $\langle a_0, \dots, a_m \rangle = p_0^{a_0+1} \dots p_m^{a_m+1}$ , and  $p_i$  is the  $i^{\text{th}}$  prime number (i.e.  $p_0 = 2, p_1 = 3, p_2 = 5, \dots$ ).

We also have the following definitions:

- $lh(a)$  is defined as the least  $n$  such that either  $a = 0$  or  $p_n$  does not divide  $a$ .
- $(a)_b$  is defined as the least  $n$  such that either  $a = 0$  or  $p_b^{n+2}$  does not divide  $a$ .
- Finally,

$$a * b = a \prod_{i < lh(b)} p_{i+lh(a)}^{(b)_i+1}.$$

- (a) Is 3 a sequence number?
- (b) What is  $lh(3)$ ?
- (c) Find  $(1 * 3) * 6$  and  $1 * (3 * 6)$ .