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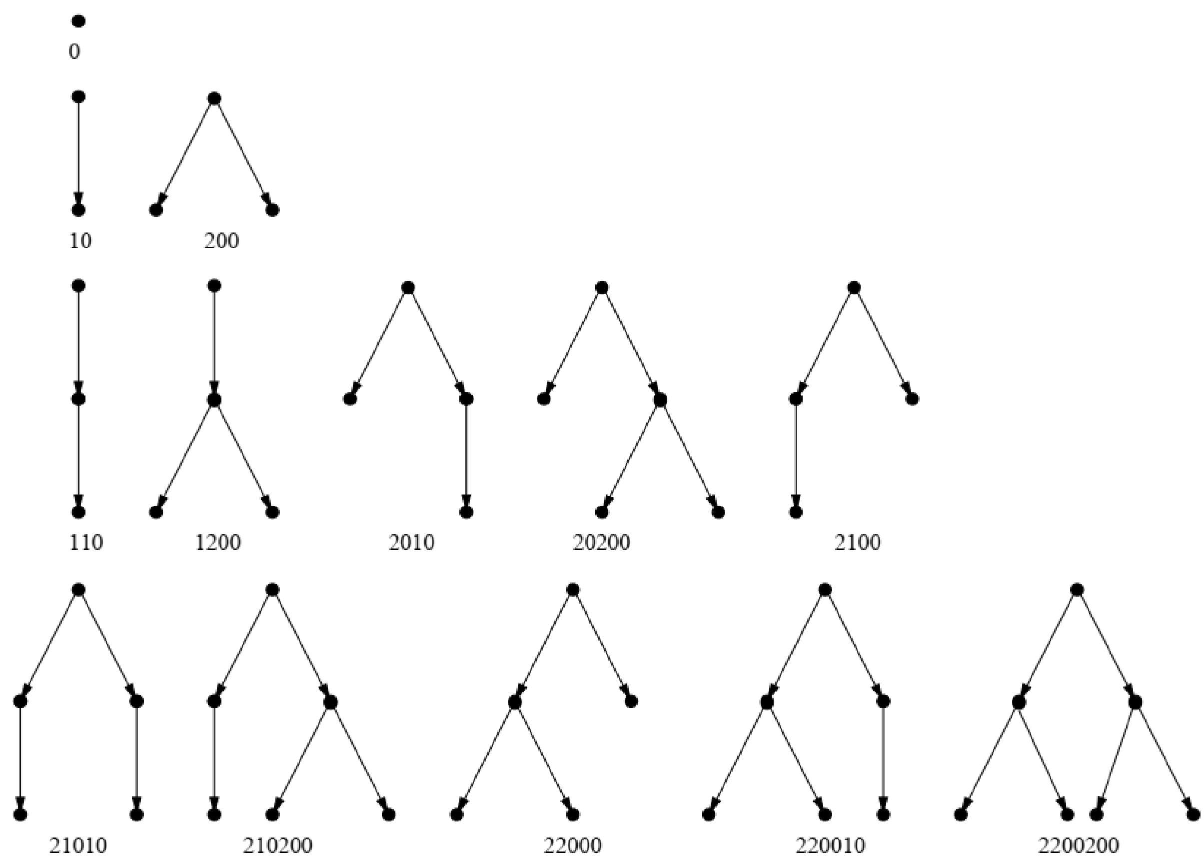
SOLUTION TO PROBLEMS

QUESTION ONE

Consider the set of directed, rooted, ordered binary trees in which any node may have 0, 1, or 2 children. These trees can be represented by expressions over the alphabet $\{0, 1, 2\}$ as follows. The tree with a single vertex is represented by the expression 0. Any other tree is represented by starting with the digit corresponding to the number of children at the root node and then concatenating the expressions corresponding to the sub-trees for each child in order. All such trees of height 2 or less are shown below with their corresponding expressions.



(a) Give an inductive definition of the set T of expressions which represent such trees.

Answer:

- $U =$ the set of expressions over $\{0, 1, 2\}$.
- $B = \{0\}$.

- F consists of two functions:

$$t_1(\alpha) = 1\alpha$$
$$t_2(\alpha, \beta) = 2\alpha\beta$$

Then T is the set generated from B by F.

(b) Prove that T is freely generated. You will need to prove a lemma similar to the one proved in class (and in the solutions to last year's problems).

Answer:

Define a function f on expressions to be the number of occurrences of 0 minus the number of occurrences of 2. Clearly $f(\alpha\beta) = f(\alpha) + f(\beta)$.

First we'll show that for any $\alpha \in T$, $f(\alpha) = 1$. Take S to be the set of expressions s such that $f(s) = 1$. The only element of B is 0, and $f(0) = 1$ by definition, so $B \subseteq S$.

Now, assume $\alpha \in S$. We have $f(t_1(\alpha)) = f(1\alpha) = f(1) + f(\alpha)$. $f(1) = 0$ by definition, and $f(\alpha) = 1$ by assumption, so $f(t_1(\alpha)) = 1$ and thus $t_1(\alpha) \in S$.

Similarly, assume $\alpha, \beta \in S$. We have $f(t_2(\alpha, \beta)) = f(2\alpha\beta) = f(2) + f(\alpha) + f(\beta)$, $f(2) = -1$ by definition, and $f(\alpha) = f(\beta) = 1$ by assumption, so $f(t_2(\alpha, \beta)) = 1$ and $t_2(\alpha, \beta) \in S$.

Therefore, by the induction principle, $T \subseteq S$, and for any $\alpha \in T$, $f(t) = 1$.

Now we'll show that for any proper initial segment α_0 of an expression $\alpha \in T$, $f(\alpha_0) \leq 0$. Take S to be the set of expressions $s \in T$ such that for any proper initial segment s_0 of s, $f(s_0) \leq 0$. There are no proper initial segments of 0, so clearly $B \subseteq S$.

Now, assume $\alpha \in S$. Say $s = t_1(\alpha) = 1\alpha$. The proper initial segments of s are 1 and $1\alpha_0$ where α_0 is a proper initial segment of α. We have $f(1) = 0$ by definition, and $f(1\alpha_0) = f(1) + f(\alpha_0) = f(\alpha_0) \leq 0$ by assumption. Therefore $t_1(\alpha) \in S$.

Similarly, assume $\alpha, \beta \in S$. Say $s = t_2(\alpha, \beta) = 2\alpha\beta$. The proper initial segments of s are 2, $2\alpha_0$, 2α, and $2\alpha\beta_0$. We have $f(2) = -1$ by definition, $f(2\alpha_0) = f(2) + f(\alpha_0) = -1 + f(\alpha_0) \leq 0$ by assumption, $f(2\alpha) = f(2) + f(\alpha) = -1 + 1 = 0$ since α is in T, and $f(2\alpha\beta_0) = f(2) + f(\alpha) + f(\beta_0) = f(\beta_0) \leq 0$. Therefore, $t_2(\alpha, \beta) \in S$.

Therefore, by the induction principle, $T \subseteq S$ and for any proper initial segment α_0 of an expression $\alpha \in T$, $f(\alpha_0) \leq 0$. It follows that no expression in T can be a proper initial segment of another.

Now we'll show that the functions in F are one-to-one, and that their ranges do not overlap. Say $\alpha, \gamma \in T$, and $t_1(\alpha) = t_1(\gamma)$. Then $1\alpha = 1\gamma$, so $\alpha = \gamma$, so t_1 is one-to-one.

Similarly, say we have α, β, γ, $\delta \in T$, and $t_2(\alpha, \beta) = t_2(\gamma, \delta)$. Then $2\alpha\beta = 2\gamma\delta$, and $\alpha\beta = \gamma\delta$. Since no element of T can be a proper initial segment of another, $\alpha = \gamma$, and $\beta = \delta$, so t_2 is one-to-one.

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It's easy to see that no functions in F have overlapping ranges, because for each function in F, every expression in the range of the function starts with a different symbol. Similarly, the ranges of the functions in F are disjoint from B because none of those symbols are 0.

Therefore T is freely generated.

(c) Prove that 1020 and 2210 are not in T.

Answer:

1020 has 10 as a proper initial segment, and $f(10) = 1$. But we showed above that $f(\alpha_0) \leq 0$ for every proper initial segment α_0 of an expression $\alpha \in T$. Therefore, $1020 \notin T$.

$f(2210) = -1$, but we showed above that $f(\alpha) = 1$ for all expressions $\alpha \in T$. Therefore, $2210 \notin T$.

(d) The *height* of such a tree is defined to be the longest possible path starting at the root node. For example, the tree 0 has height 0. The trees 10 and 200 have height 1, and the rest of the trees shown above have height 2. Give a recursive definition of a function which returns the height of any tree in T.

Answer:

$h(0) = 0$
$h(t_1(\alpha)) = 1 + h(\alpha)$
$h(t_2(\alpha, \beta)) = 1 + \max(h(\alpha), h(\beta))$

QUESTION TWO (FOUR PARTS)

1. Prove that for all *wff*'s α, α is satisfiable iff $\neg\alpha$ is not valid.

Answer:

$$\begin{aligned} \alpha \text{ is satisfiable} \quad & \text{iff} \quad \overline{v}(\alpha) = \mathbf{T} \text{ for some truth assignment } v \\ & \text{iff} \quad \mathbf{T} - \overline{v}(\alpha) = \mathbf{F} \text{ for some truth assignment } v \\ & \text{iff} \quad \overline{v}(\neg\alpha) = \mathbf{F} \text{ for some truth assignment } v \\ & \text{iff} \quad \neg\alpha \text{ is not valid.} \end{aligned}$$

2. Is the following formula valid? Justify your answer using a truth table:

$$(((P \wedge Q) \rightarrow R) \rightarrow S) \rightarrow (((R \rightarrow Q) \rightarrow P) \rightarrow S).$$

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Answer:

The formula is not valid.

P	Q	R	S	$(((P$	$\wedge$	$Q)$	$\rightarrow$	$R)$	$\rightarrow$	$S)$	$\rightarrow$	$(((R$	$\rightarrow$	$Q)$	$\rightarrow$	$P)$	$\rightarrow$	$S)$
F	F	F	F	F	F	F	T	F	F	F	T	F	T	F	F	F	T	F
F	F	F	T	F	F	F	T	F	T	T	T	F	T	F	F	F	T	T
F	F	T	F	F	F	F	T	T	F	F	T	T	F	F	T	F	F	F
F	F	T	T	F	F	F	T	T	T	T	T	T	F	F	T	F	T	T
F	T	F	F	F	F	T	T	F	F	F	T	F	T	T	F	F	T	F
F	T	F	T	F	F	T	T	F	T	T	T	F	T	T	F	F	T	T
F	T	T	F	F	F	T	T	T	F	F	T	T	T	T	F	F	T	F
F	T	T	T	F	F	T	T	T	T	T	T	T	T	T	F	F	T	T
T	F	F	F	T	F	F	T	F	F	F	T	F	T	F	T	T	F	F
T	F	F	T	T	F	F	T	F	T	T	T	F	T	F	T	T	T	T
T	F	T	F	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
T	F	T	T	T	F	F	T	T	T	T	T	T	F	F	T	T	T	T
T	T	F	F	T	T	T	F	F	T	F	F	F	T	T	T	T	F	F
T	T	F	T	T	T	T	F	F	T	T	T	F	T	T	T	T	T	T
T	T	T	F	T	T	T	T	T	F	F	T	T	T	T	T	T	F	F
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T

3. Show that $\{\downarrow\}$ is complete.

Answer:

We have $\neg\alpha$ is tautologically equivalent to $\alpha \downarrow \alpha$, and $\alpha \wedge \beta$ is tautologically equivalent to $\neg\alpha \downarrow \neg\beta$. Since $\{\neg, \wedge\}$ is complete, we need to show that for any formula α using only the connectives $\neg$ and $\wedge$, there is a tautologically equivalent formula α' using only the connective $\downarrow$.

We'll proceed by induction on the structure of α. In the base case, $\alpha = A_k$, and $\alpha' = A_k$ which is clearly tautologically equivalent.

In the case that $\alpha = \neg\beta$, assume that there is a tautologically equivalent β' using only $\downarrow$. Then let $\alpha' = \beta' \downarrow \beta'$, which is tautologically equivalent to $\beta \downarrow \beta$, which is tautologically equivalent to $\neg\beta = \alpha$, so α' is tautologically equivalent to α.

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In the case that $\alpha = \beta \wedge \gamma$, assume that there are tautologically equivalent β', γ' using only $\downarrow$. Then let $\alpha' = (\beta' \downarrow \beta') \downarrow (\gamma' \downarrow \gamma')$, which is tautologically equivalent to $(\beta \downarrow \beta) \downarrow (\gamma \downarrow \gamma)$, which is tautologically equivalent to $\neg\beta \downarrow \neg\gamma$, which is tautologically equivalent to $\beta \wedge \gamma = \alpha$, so α' is tautologically equivalent to α.

4. Say that a set Σ_1 of wffs is *equivalent* to a set Σ_2 of wffs iff for any wff α, we have $\Sigma_1 \models \alpha$ iff $\Sigma_2 \models \alpha$. A set Σ is *independent* iff no member of Σ is tautologically implied by the remaining members in Σ. Prove that every finite set of wffs has an independent equivalent subset.

Answer:

We want to show that for all $n \geq 0$, if Σ is a set of n wffs, then Σ has an independent equivalent subset. The proof is by ordinary induction on n.

Base case: $n = 0$. In this case, $\Sigma = \emptyset$. Σ is independent (vacuously), and therefore Σ is itself an independent equivalent subset.

Inductive case: $n = k+1$. Assume that every set containing k elements has an independent equivalent subset. Consider a set Σ with $k+1$ elements. If Σ is independent, then it is itself an independent equivalent subset. Otherwise, there must exist some $\phi \in \Sigma$ such that $\Sigma - \{\phi\} \models \phi$.

We claim that $\Sigma - \{\phi\}$ is equivalent to Σ.

Proof:

$\Rightarrow$ Suppose $\Sigma - \{\phi\} \models \alpha$. Let v be a truth assignment satisfying Σ. We must show that v satisfies α. If v satisfies Σ, then certainly v satisfies $\Sigma - \{\phi\}$. But then v must satisfy α by our hypothesis.

$\Leftarrow$ Suppose $\Sigma \models \alpha$. Let v be a truth assignment satisfying $\Sigma - \{\phi\}$. We must show that v satisfies α. Now, ϕ was chosen as the element for which $\Sigma - \{\phi\} \models \phi$. Thus, since v satisfies $\Sigma - \{\phi\}$, it must also satisfy ϕ. So in fact, v satisfies $\Sigma - \{\phi\} \cup \{\phi\} = \Sigma$. But by our hypothesis it then follows that v satisfies α.

We just showed that $\Sigma - \{\phi\}$ is equivalent to Σ. But $\Sigma - \{\phi\}$ has k elements, so by the induction hypothesis it has an independent equivalent subset. But $\Sigma - \{\phi\}$ is an equivalent subset of Σ, so Σ also has an independent equivalent subset.

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QUESTION THREE (TWO PARTS)

1. Prove that if ϕ and ψ are *first-order wffs*, then $\phi \models\!\!\!\dashv \psi$ iff $\models (\phi \leftrightarrow \psi)$.

   Answer:

   $\Rightarrow$ Suppose $\phi \models\!\!\!\dashv \psi$. Let M be an arbitrary model and s an arbitrary variable assignment. If $\models_M \phi[s]$, then by hypothesis, we also have $\models_M \psi[s]$. Thus, by the semantics of $\models$ for Boolean reasoning, it follows that $\models_M (\phi \leftrightarrow \psi)[s]$. On the other hand, if $\not\models_M \phi[s]$, then by using the contrapositive of (one part of) the hypothesis, we also have $\not\models_M \psi[s]$. Thus, as before, $\models_M (\phi \leftrightarrow \psi)[s]$. Since M and s were chosen arbitrarily, $\models (\phi \leftrightarrow \psi)$.

   $\Leftarrow$ Suppose $\models (\phi \leftrightarrow \psi)$. Now, suppose M is a model and s a variable assignment such that $\models_M \phi[s]$. Since we know that $\models_M (\phi \leftrightarrow \psi)[s]$, it must be the case (by definition of $\models$ for the Boolean operators that make up $\leftrightarrow$) that $\models_M \psi[s]$, so $\phi \models \psi$. A similar argument holds in the other direction. Thus, $\phi \models\!\!\!\dashv \psi$.

2. Suppose P is a binary predicate. Show that no one of the following sentences is logically implied by the other two. Do this by giving a model for each sentence in which the sentence is false but the other two sentences are true.

   (a) $\forall x\, Pxx$

   (b) $\forall x \forall y\, (Pxy \lor Pyx \lor x = y)$

   (c) $\exists x \forall y\, Pxy$

   Answer:

   Consider a model M with $dom(M) = \{a, b, c\}$.

   (a),(b),¬(c): $P^M = \{(a,a), (b,b), (c,c), (a,b), (b,c), (c,a)\}$

   (a),¬(b),(c): $P^M = \{(a,a), (b,b), (c,c), (a,b), (a,c)\}$

   ¬(a),(b),(c): $P^M = \{(a,a), (a,b), (a,c), (b,c)\}$

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QUESTION FOUR (TWO PARTS)

1.

Consider a language with equality and a single binary predicate symbol P. For each set $\mathcal{M}$ of models below, write a first order sentence ϕ such that $\models_M \phi$ iff $M \in \mathcal{M}$.

(a) $\mathcal{M} = \{M | P^M$ is a transitive relation $\}$.

Answer: $\forall x \forall y \forall z\, ((Pxy \wedge Pyz) \rightarrow Pxz)$

(b) $\mathcal{M} = \{M | P^M$ defines a function $\}$.

Answer: $\forall x \forall y \forall z\, ((Pxy \wedge Pxz) \rightarrow y = z) \wedge \forall x \exists y\, Pxy$

(c) $\mathcal{M} = \{M | P^M$ is a bijection (i.e. a function that is 1-1 and onto) $\}$.

Answer: $\forall x \forall y \forall z\, ((Pxy \wedge Pxz) \rightarrow y = z) \wedge \forall x \exists y\, Pxy \wedge$
$(\forall x \forall y \forall z\, ((Pxy \wedge Pzy) \rightarrow x = z) \wedge \forall y \exists x\, Pxy$

2.

Consider a signature Σ with no constant symbols, no predicate symbols (except for equality), and a single binary function symbol, $+$. Let M be a Σ-model with domain $\mathbf{N}$ (the natural numbers) which interprets $+$ in the standard way.

Note that the only non-logical symbols you may use are $=$ and $+$.

(a) Give a Σ-formula which defines the set $\{0\}$ in M.

Answer:

$v_1 + v_1 = v_1$

(b) Give a Σ-formula which defines the set $\{1\}$ in M.

Answer:

$\forall v_2\, (v_2 + v_2 = v_2 \rightarrow (v_2 \neq v_1 \wedge \forall v_3\, (v_3 = v_2 \vee \exists v_4\, (v_3 = v_1 + v_4))))$

(c) Give a Σ-formula which defines the binary relation $\{\langle m, n \rangle \,| m < n\}$ in M.

Answer:

$\forall v_3\, (v_3 + v_3 = v_3 \rightarrow \exists v_4\, (v_1 + v_4 = v_2 \wedge v_3 \neq v_4))$

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QUESTION FIVE (THREE PARTS)

1. Give a deduction (from $\emptyset$) of $\forall x\, Pxx \to y = z \to Pyz$.

   Answer:

1.	$\forall x\, Pxx \to Pyy$	axiom group 2
2.	$y = z \to Pyy \to Pyz$	axiom group 6
3.	$(\forall x\, Pxx \to Pyy) \to$ $(y = z \to Pyy \to Pyz) \to (\forall x\, Pxx \to y = z \to Pyz)$	tautology: $(A \to B) \to (C \to B \to D) \to (A \to C \to D)$
4.	$(y = z \to Pyy \to Pyz) \to (\forall x\, Pxx \to y = z \to Pyz)$	modus ponens 1,3
5.	$\forall x\, Pxx \to y = z \to Pyz$	modus ponens 2,4

2. Assume that x does not occur free in α. Show that

$$\vdash (\alpha \to \exists x\, \beta) \leftrightarrow \exists x\, (\alpha \to \beta)$$

   (i.e. show $\vdash (\alpha \to \exists x\, \beta) \to \exists x\, (\alpha \to \beta)$ and $\vdash \exists x\, (\alpha \to \beta) \to (\alpha \to \exists x\, \beta)$). Note that you do not have to give the deduction, just prove that one exists.

   Answer:

   To show

$$\vdash (\alpha \to \exists x\, \beta) \to \exists x\, (\alpha \to \beta),$$

   by the deduction theorem, it suffices to show that

$$(\alpha \to \exists x\, \beta) \vdash \exists x\, (\alpha \to \beta),$$

   which (expanding $\exists$) is equivalent to

$$(\alpha \to \neg\forall x\, \neg\beta) \vdash \neg\forall x\, \neg(\alpha \to \beta),$$

   which (by contraposition) is equivalent to

$$\forall x\, \neg(\alpha \to \beta) \vdash \neg(\alpha \to \neg\forall x\, \neg\beta).$$

   Now, by rule T (since $\{A, B\}$ tautologically implies $\neg(A \to \neg B)$), it suffices to show that

$$\forall x\, \neg(\alpha \to \beta) \vdash \alpha, \text{ and}$$

$$\forall x\, \neg(\alpha \to \beta) \vdash \forall x\, \neg\beta.$$

   The first follows by axiom 2 and the tautology $\neg(A \to B) \to A$. For the second, note that the same argument (with tautology $\neg(A \to B) \to \neg B$) shows that $\forall x\, \neg(\alpha \to \beta) \vdash \neg\beta$. It follows by generalization (since x is not free on the left side) that $\forall x\, \neg(\alpha \to \beta) \vdash \forall x\, \neg\beta$.

   For the other direction, to show

$$\vdash \exists x\, (\alpha \to \beta) \to (\alpha \to \exists x\, \beta)$$

   it suffices (by the deduction theorem, applied twice) to show that

$$\{\exists x\,(\alpha \to \beta), \alpha\} \vdash \exists x\,\beta,$$

which (expanding $\exists$) is equivalent to

$$\{\neg\forall x\,\neg(\alpha \to \beta), \alpha\} \vdash \neg\forall x\,\neg\beta,$$

which (by contraposition) is equivalent to

$$\{\forall x\,\neg\beta, \alpha\} \vdash \forall x\,\neg(\alpha \to \beta).$$

Now, clearly $\{\forall x\,\neg\beta, \alpha\} \vdash \alpha$, and by axiom 2, $\{\forall x\,\neg\beta, \alpha\} \vdash \neg\beta$. Thus, by rule T (since $\{A, \neg B\}$ tautologically implies $\neg(A \to B)$), we have

$$\{\forall x\,\neg\beta, \alpha\} \vdash \neg(\alpha \to \beta)$$

Finally, since x does not appear free in the left hand side (we know this because we assumed x was not free in α), we have by generalization,

$$\{\forall x\,\neg\beta, \alpha\} \vdash \forall x\,\neg(\alpha \to \beta).$$

3. A complete calculus has the property that each sentence either has a deduction (from $\emptyset$) or a counter-model (i.e., a model in which it is false). For each of the following sentences, either show there is a deduction or give a counter-model.

(a) $\forall x\,(Qx \to \forall y\,Qy)$

Answer:

Consider a model M with $dom(M) = \{a, b\}$ and $Q^M = \{a\}$. Then Qx is true when $x = a$, but $\forall y\,Qy$ is false.

(b) $\forall z\,(Pz \to Qz) \to (\exists x\,Px \to \forall y\,Qy)$

Answer:

Consider a model M with $dom(M) = \{a, b\}$, $P^M = \{a\}$, and $Q^M = \{a\}$. Then $\forall z(Pz \to Qz)$ is true since Q is true whenever P is. Also, $\exists x\,Px$ is true since P is true for a. But $\forall y\,Qy$ is false since Q does not hold for b.

(c) $\neg\exists y\,\forall x\,(Pxy \leftrightarrow \neg Pxx)$

Assume that $A \leftrightarrow B$ is an abbreviation for $\neg((A \to B) \to \neg(B \to A))$.

Answer:

$$\vdash \neg\exists y\,\forall x\,(Pxy \leftrightarrow \neg Pxx)$$

is equivalent (rewriting $\exists$) to

$$\vdash \neg\neg\forall y\neg\,\forall x\,(Pxy \leftrightarrow \neg Pxx).$$

To show this, it suffices (by the tautology $A \to \neg\neg A$) to show

$$\vdash \forall y\neg\,\forall x\,(Pxy \leftrightarrow \neg Pxx).$$

To show this, it suffices (by generalization) to show

$$\vdash \neg\,\forall x\,(Pxy \leftrightarrow \neg Pxx).$$

Now, using the substitution strategy (see slide 24 of lecture 5 or point 3c on p. 121 of the book), substituting y for x, it suffices to show that

$$\vdash \neg(Pyy \leftrightarrow \neg Pyy).$$

But this is a tautology, so we are done.

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QUESTION SIX (TWO PARTS)

1. Show that the following sentence is finitely valid (i.e. true in every finite model). Hint: show that any model of the negation must be infinite.

$$\exists x \forall y \exists z \; [(Qzx \to Qzy) \to (Qxy \to Qxx)]$$

Answer:

One version of the negation of the sentence is:

$$\forall x \exists y \, (Qxy \wedge \forall z \, (Qzx \to Qzy)) \wedge \forall x \, \neg Qxx.$$

Suppose M is a model of the sentence, and consider the directed graph G whose vertices are elements of M and that has an edge from a to b iff $Q^M(a, b)$. From the formula we see that:

i. If a is a vertex, then for some vertex b, there is an edge from a to b and whenever there is an edge to a from another vertex c, there is also an edge to b from c.

ii. No vertex has a self-loop

Now, let $v_1, v_2, \ldots$ be a path through the graph defined as follows:

- v_1 is an arbitrary vertex in the graph.
- v_{i+1} is the vertex that must exist by property (i) applied to v_i: i.e. there is an edge from v_i to v_{i+1} and if any other vertex has an edge to v_i it also has an edge to v_{i+1}.

We claim that if $j > i$, then there is an edge from v_i to v_j. Proof: Let $j = i + k$. We will prove the claim by induction on k. For $k = 1$, we know that there is an edge from v_i to v_{i+1} by definition. Now, suppose that there is an edge from v_i to v_{i+k} and consider v_{i+k+1}. We know there is an edge from v_{i+k} to v_{i+k+1} and from v_i to v_k. Thus, by property (i) and the way we constructed the path, there must be an edge from v_i to v_{i+k+1}.

Finally, suppose the model (and therefore also the graph) is finite. Then, since there are only a finite number of vertices, we must eventually have $v_i = v_{i+k}$ for some $i, k > 0$. But by the above claim, this implies that there is an edge from v_i to itself which contradicts (ii), above. Thus, the model cannot be finite.

2. Let ϕ be the sentence $\forall x \forall y \forall z \, (Pxy \to Pyz \to Pxz)$. Recall that a theory is (the deductive closure of) a set of sentences. Give an example of a theory T such that

(a) ϕ is T-valid,

Answer:

Let $T = Cn \, \phi$. Clearly, ϕ is true in every model of T, so ϕ is T-valid.

(b) ϕ is T-satisfiable but not T-valid,

Answer:

Let $T = Cn \, \emptyset$. T does not say anything about P. Thus, there are models of T in which P is transitive and models of T in which P is not transitive.

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(c) ϕ is T-unsatisfiable

Answer:

Let $T = Cn\neg\phi$. Clearly ϕ is false in every model of T.

QUESTION SEVEN (TWO PARTS)

1.

Assume that Augustine is loved by everyone who loves someone. Also assume that no one loves nobody. We wish to deduce that Augustine is loved by everyone.

(a) Consider a language with one constant symbol a (for Augustine) and a 2-place predicate symbol L, so that $L(x, y)$ encodes "x loves y". Write three first-order sentences which encode the two premises and the conclusion.

Answer:

i. $\forall x\, (\exists y\, L(x, y) \rightarrow L(x, a))$

ii. $\neg\exists x\, \neg\exists y\, L(x, y)$

iii. $\forall x\, L(x, a)$

(b) To show that the premises imply the conclusion, we will show that it is impossible to satisfy the premises and the negation of the conclusion. Skolemize each premise and the negation of the conclusion to obtain three clauses.

Answer:

We first write the sentences in prenex form:

i. $\forall x\, \forall y\, (L(x, y) \rightarrow L(x, a))$

ii. $\forall x\, \exists y\, L(x, y)$

iii. $\exists x\, \neg L(x, a)$

Skolemizing, we get:

i. $\forall x\, \forall y\, (L(x, y) \rightarrow L(x, a))$

ii. $\exists F\, \forall x\, L(x, Fx)$

iii. $\exists c\, \neg L(c, a)$

Note that c is a "0-ary" function which is the same as a constant. Dropping the quantifiers, we get the following three clauses:

i. $(\neg L(x, y) \vee L(x, a))$

ii. $(L(x, Fx))$

iii. $(\neg L(c, a))$

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(c) Use first-order resolution to show that the empty clause is obtainable from the original three clauses.

Answer:

Resolving (i) and (ii) using the substitution $y \leftarrow Fx$ yields:

$$(L(x, a))$$

Resolving this with (iii) using the substitution $x \leftarrow c$ yields the empty clause.

Alternatively, resolving (i) with (iii) using the substitution $x \leftarrow c$ yields:

$$(\neg L(c, y))$$

Resolving this with (ii) using the substitution $x \leftarrow c, y \leftarrow Fc$ yields the empty clause.

2.

Recall that b is a *sequence number* iff for some $m \geq -1$ and some $a_0, \ldots, a_m$, $b = \langle a_0, \ldots, a_m \rangle$, where $\langle a_0, \ldots, a_m \rangle = p_0^{a_0+1} \cdots p_m^{a_m+1}$, and p_i is the i^{th} prime number (i.e. $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$).

We also have the following definitions:

- $lh(a)$ is defined as the least n such that either $a = 0$ or p_n does not divide a.
- $(a)_b$ is defined as the least n such that either $a = 0$ or p_b^{n+2} does not divide a.
- Finally,

$$a * b = a \prod_{i < lh(b)} p_{i+lh(a)}^{(b)_i + 1}.$$

(a) Is 3 a sequence number?

(b) What is $lh(3)$?

(c) Find $(1 * 3) * 6$ and $1 * (3 * 6)$.